Write your name here

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| MODEL SOLUTIONS | |

**Pearson Edexcel Level 3 GCE**

| Centre Number | Candidate Number |
| --- | --- |
| | |

# Mathematics

## Advanced
## Paper 1: Pure Mathematics 1

| Specimen Paper | Paper Reference |
| --- | --- |
| **Time: 2 hours** | **9MA0/01** |

**You must have:**
Mathematical Formulae and Statistical Tables, calculator

Total Marks

**Candidates may use any calculator allowed by the regulations of the Joint Council for Qualifications. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.**

## Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need*.
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Answers should be given to three significant figures unless otherwise stated.

## Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 14 questions in this question paper. The total mark for this paper is 100.
- The marks for each question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

## Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end

*Turn over* ▶









Pearson

**Answer ALL questions. Write your answers in the spaces provided.**

**1.**

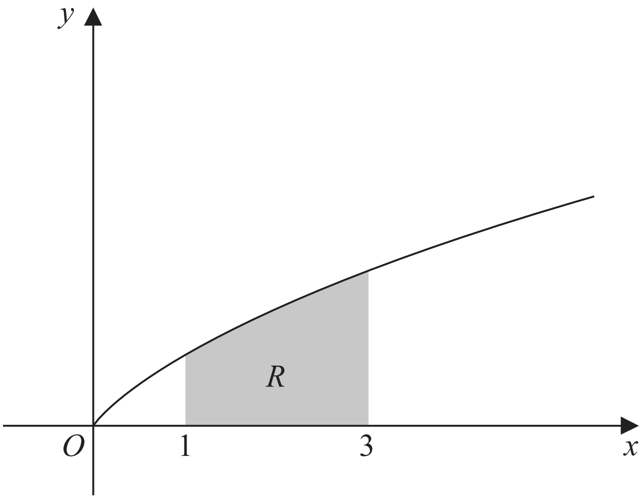


**Figure 1**

Figure 1 shows a sketch of the curve with equation $y = \dfrac{x}{1+\sqrt{x}}, \quad x \geqslant 0$

The finite region $R$, shown shaded in Figure 1, is bounded by the curve, the line with equation $x = 1$, the $x$-axis and the line with equation $x = 3$

The table below shows corresponding values of $x$ and $y$ for $y = \dfrac{x}{1+\sqrt{x}}$

| $x$ | 1 | 1.5 | 2 | 2.5 | 3 |
|---|---|---|---|---|---|
| $y$ | 0.5 | 0.6742 | 0.8284 | 0.9686 | 1.0981 |

$y_0$ $y_1$ $y_2$ $y_3$ $y_4$

(a) Use the trapezium rule, with all the values of $y$ in the table, to find an estimate for the area of $R$, giving your answer to 3 decimal places.

**(3)**

(b) Explain how the trapezium rule can be used to give a better approximation for the area of $R$.

**(1)**

(c) Giving your answer to 3 decimal places in each case, use your answer to part (a) to deduce an estimate for

(i) $\displaystyle\int_1^3 \frac{5x}{1+\sqrt{x}}\,\mathrm{d}x$

(ii) $\displaystyle\int_1^3 \left(6 + \frac{x}{1+\sqrt{x}}\right)\mathrm{d}x$

**(2)**

**Question 1 continued**

a)

$$y = \frac{x}{1+\sqrt{x}}$$

$$\int_a^b y\,dx \approx \frac{1}{2}h\left\{y_0 + y_n + 2(y_1 + y_2 \ldots y_{n-1})\right\}$$

$$h = \frac{b-a}{n} \qquad \therefore a = 1 \quad n = 4 \quad b = 3$$

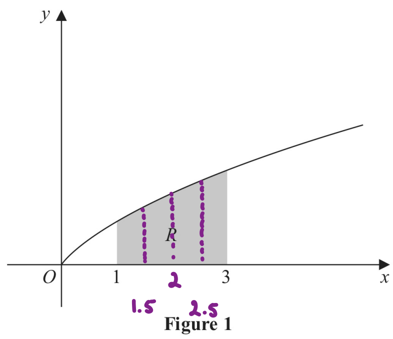

Figure 1

$$\int_1^3 \frac{x}{1+\sqrt{x}}\,dx \approx \frac{3-1}{4} \times \frac{1}{2}\left\{0.5 + 1.0981 + 2(0.6742 + 0.8284 + 0.9686)\right\}$$

$$\approx \frac{1}{4} \times 6.5405$$

$$= 1.635125 = 1.635\ (3.d.p).$$

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**Question 1 continued**

b)

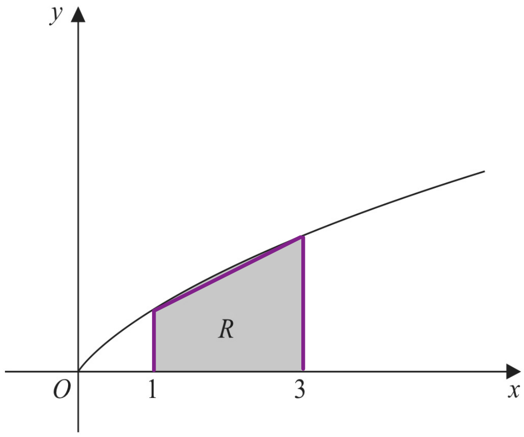



**Figure 1**

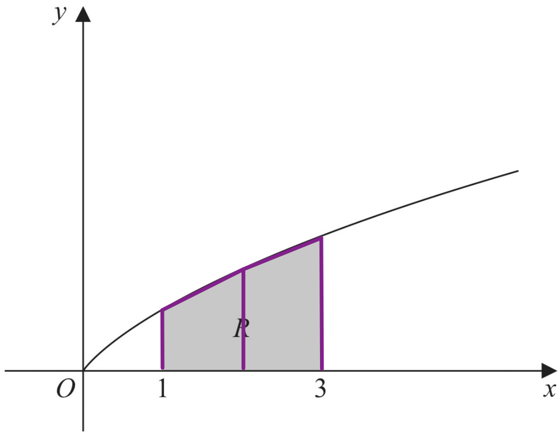



**Figure 1**

Increase the number of strips.

**Question 1 continued**

c)

$$\int_1^3 \frac{5x}{1+\sqrt{x}}\,dx = 5\int_1^3 \frac{x}{1+\sqrt{x}}\,dx \approx 5 \times 1.635 = 8.175$$

$$\int_1^3 \left(6 + \frac{x}{1+\sqrt{x}}\right) dx = [6x]_1^3 + \int_1^3 \frac{x}{1+\sqrt{x}}\,dx$$

$$\int_1^3 \frac{x}{1+\sqrt{x}}\,dx \approx 1.635$$

$$[6x]_1^3 + \int_1^3 \frac{x}{1+\sqrt{x}}\,dx \approx [(6\times 3)-(6\times 1)] + 1.635$$

$$= 12 + 1.635 = 13.635$$

**(Total for Question 1 is 6 marks)**

**2.** (a) Show that the binomial expansion of

$$(4 + 5x)^{\frac{1}{2}}$$

in ascending powers of $x$, up to and including the term in $x^2$ is

$$2 + \frac{5}{4}x + kx^2$$

giving the value of the constant $k$ as a simplified fraction.

**(4)**

(b) (i) Use the expansion from part (a), with $x = \frac{1}{10}$, to find an approximate value for $\sqrt{2}$

Give your answer in the form $\frac{p}{q}$ where $p$ and $q$ are integers.

(ii) Explain why substituting $x = \frac{1}{10}$ into this binomial expansion leads to a valid approximation.

**(4)**

**Question 2 continued**

a)

$(1+x)^n = 1 + nx + \frac{(n)(n-1)x^2}{2}$
$(n \in \mathbb{Q})$

$(a+bx)^n = a^n\left(1+\frac{bx}{a}\right)^n$

$(4+5x)^{1/2} = 4^{1/2}\left(1+\frac{5}{4}x\right)^{1/2}$

$= 2\left(1+\frac{5}{4}x\right)^{1/2}$ Since $n = \frac{1}{2}$,

$2\left(1+\frac{5}{4}x\right)^{1/2} = 2\left(1+\frac{5}{8}x + \frac{(0.5)(-0.5)}{2} \times \frac{25}{16}x^2\right)$

$= 2\left(1+\frac{5}{8}x - \frac{25}{128}x^2\right)$

$= 2 + \frac{5}{4}x - \frac{25}{64}x^2$

$\therefore k = -\frac{25}{64}$

**Question 2 continued**

b)

$(4+5x)^{1/2} \approx 2 + \frac{5}{4}x - \frac{25}{64}x^2$

$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

i) $\left(4+\frac{1}{2}\right)^{1/2} = (4.5)^{1/2} = \left(\frac{9}{2}\right)^{1/2} = \frac{3}{\sqrt{2}}$

$2 + \frac{5}{4} \times \frac{1}{10} - \frac{25}{64} \times \left(\frac{1}{10}\right)^2 = \frac{543}{256}$

$\frac{3}{\sqrt{2}} = \frac{543}{256}$

$\frac{3}{\sqrt{2}} \times \frac{1}{3} = \frac{1}{\sqrt{2}}$

$\frac{1}{\sqrt{2}} \times 2 = \frac{2}{\sqrt{2}} = \sqrt{2}$

$\frac{3}{\sqrt{2}} \times \frac{1}{3} \times 2 = \sqrt{2}$

$\frac{3}{\sqrt{2}} \times \frac{2}{3} = \sqrt{2}$

$\frac{3}{\sqrt{2}} \times \frac{2}{3} = \frac{543}{256} \times \frac{2}{3}$

$\sqrt{2} = \frac{181}{128}$

ii) $(p+qx)^n$, valid when $\left|\frac{qx}{p}\right| < 1$ $\quad |qx| < |p|$ $\quad |x| < \frac{|p|}{|q|}$

$(4+5x)^{1/2}$ valid when $|x| < \frac{4}{5}$

$\frac{1}{10} < \frac{4}{5}$ $\therefore$ approximation is valid.

**(Total for Question 2 is 8 marks)**

**3.** A sequence of numbers $a_1, a_2, a_3, \ldots$ is defined by

$$a_1 = 3$$

$$a_{n+1} = \frac{a_n - 3}{a_n - 2}, \qquad n \in \mathbb{N}$$

(a) Find $\sum_{r=1}^{100} a_r$

**(3)**

(b) Hence find $\sum_{r=1}^{100} a_r + \sum_{r=1}^{99} a_r$

**(1)**

a) $a_1 = 3$, $a_2 = \frac{3-3}{3-2} = 0$, $a_3 = \frac{0-3}{0-2} = 1.5$

$a_4 = \frac{1.5-3}{1.5-2} = 3$ ✓

Sequence repeats after every 3 terms.

$\sum_{r=1}^{100} a_r = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + \ldots + a_{97} + a_{98} + a_{99} + a_{100}$

$= (a_1 + a_2 + a_3) + (a_1 + a_2 + a_3) + \ldots + (a_1 + a_2 + a_3) + a_4$

$= 33(a_1 + a_2 + a_3) + a_1$

$= 33(3 + 0 + 1.5) + 3$

$= 33(4.5) + 3$ ✓ $= 151.5$ ✓

**Question 3 continued**

b) $\sum_{r=1}^{100} a_r = 151.5$

$\sum_{r=1}^{99} a_r = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + \ldots + a_{97} + a_{98} + a_{99}$

$= (a_1 + a_2 + a_3) + (a_1 + a_2 + a_3) + \ldots + (a_1 + a_2 + a_3)$

$= 33(a_1 + a_2 + a_3) = 33(4.5) = 148.5.$

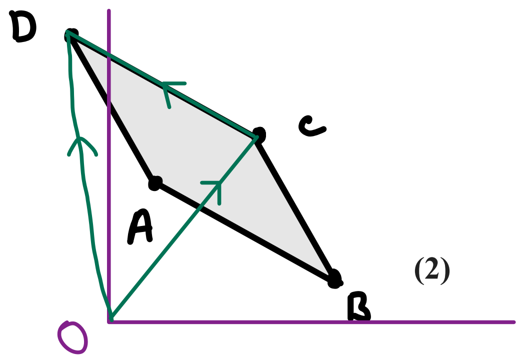
$\sum_{r=1}^{100} a_r + \sum_{r=1}^{99} a_r = 151.5 + 148.5 = 300$ ✓

**(Total for Question 3 is 4 marks)**

**4.** Relative to a fixed origin $O$,

$\overrightarrow{CD} = \overrightarrow{BA}$

the point $A$ has position vector $\mathbf{i} + 7\mathbf{j} - 2\mathbf{k}$,
the point $B$ has position vector $4\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$,
and the point $C$ has position vector $2\mathbf{i} + 10\mathbf{j} + 9\mathbf{k}$.




Given that $ABCD$ is a parallelogram,

(a) find the position vector of point $D$. **(2)**

$\hookrightarrow \overrightarrow{OD}$

The vector $\overrightarrow{AX}$ has the same direction as $\overrightarrow{AB}$.

Given that $|\overrightarrow{AX}| = 10\sqrt{2}$,

(b) find the position vector of $X$. **(3)**

$\hookrightarrow \overrightarrow{OX}$

a) $\overrightarrow{OA} = \begin{pmatrix} 1 \\ 7 \\ -2 \end{pmatrix}$ $\quad \overrightarrow{OB} = \begin{pmatrix} 4 \\ 3 \\ 3 \end{pmatrix}$ $\quad \overrightarrow{OC} = \begin{pmatrix} 2 \\ 10 \\ 9 \end{pmatrix}$

$\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD}$ $\quad$ Since $\overrightarrow{CD} = \overrightarrow{BA}$

$\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{BA}$

$= \begin{pmatrix} 2 \\ 10 \\ 9 \end{pmatrix} + \begin{pmatrix} -3 \\ 4 \\ -5 \end{pmatrix}$ ✓

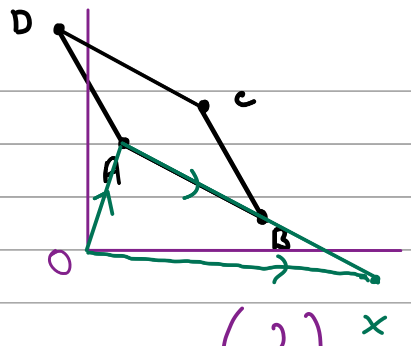
$\overrightarrow{BA} = -\overrightarrow{OB} + \overrightarrow{OA}$

$= \begin{pmatrix} -4 \\ -3 \\ -3 \end{pmatrix} + \begin{pmatrix} 1 \\ 7 \\ -2 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \\ -5 \end{pmatrix}$

$= \begin{pmatrix} -1 \\ 14 \\ 4 \end{pmatrix}$

$\therefore \overrightarrow{OD} = -\underline{i} + 14\underline{j} + 4\underline{k}$ ✓

**Question 4 continued**




$$\overrightarrow{OA} = \begin{pmatrix} 1 \\ 7 \\ -2 \end{pmatrix} \quad \overrightarrow{OB} = \begin{pmatrix} 4 \\ 3 \\ 3 \end{pmatrix} \quad \overrightarrow{OC} = \begin{pmatrix} 2 \\ 10 \\ 9 \end{pmatrix}$$

$$\overrightarrow{AB} = -\overrightarrow{OA} + \overrightarrow{OB} = \begin{pmatrix} -1 \\ -7 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix}$$

$$|\overrightarrow{AB}| = \sqrt{(3)^2 + (-4)^2 + (5)^2} = \sqrt{9+16+25} = \sqrt{50}$$
$$= \sqrt{25} \times \sqrt{2}$$
$$= 5\sqrt{2}$$

Since $|\overrightarrow{AB}| = 5\sqrt{2}$, $|\overrightarrow{AX}| = 2 \times |\overrightarrow{AB}|$

$$\overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AX} \quad \therefore \overrightarrow{OX} = \overrightarrow{OA} + 2\overrightarrow{AB}$$

$$= \begin{pmatrix} 1 \\ 7 \\ -2 \end{pmatrix} + 2\begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 7 \\ -2 \end{pmatrix} + \begin{pmatrix} 6 \\ -8 \\ 10 \end{pmatrix}$$

$$= \begin{pmatrix} 7 \\ -1 \\ 8 \end{pmatrix}$$

$$\therefore \overrightarrow{OX} = 7\underline{i} - \underline{j} + 8\underline{k}$$

**(Total for Question 4 is 5 marks)**

**5.** $\mathrm{f}(x) = x^3 + ax^2 - ax + 48$, where $a$ is a constant

Given that $\mathrm{f}(-6) = 0$

(a) (i) show that $a = 4$

(ii) express $\mathrm{f}(x)$ as a product of two algebraic factors.

**(4)**

Given that $2\log_2(x + 2) + \log_2 x - \log_2(x - 6) = 3$

(b) show that $x^3 + 4x^2 - 4x + 48 = 0$

**(4)**

(c) hence explain why

$$2\log_2(x + 2) + \log_2 x - \log_2(x - 6) = 3 \Rightarrow f(x) = 0$$

has no real roots.

**(2)**

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S 6 0 7 3 6 A 0 1 0 3 2

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**Question 5 continued**

a)

i) if $f(-6)=0$, $(-6)^3 + a(-6)^2 - a(-6) + 48 = 0$

$-216 + 36a + 6a + 48 = 0$

$42a = 168$

$a = \frac{168}{42} = 4$

$\therefore a = 4$ as required.

ii) $f(x) = x^3 + 4x^2 - 4x + 48$

if $f(a) = 0$, $x - a$ is a factor of $f(x)$.

$f(-6) = 0$ $\therefore x+6$ is a factor of $f(x)$.

$$
\begin{array}{r|l}
 & x^2 - 2x + 8 \\
\hline
x+6 & x^3 + 4x^2 - 4x + 48 \\
 & -(x^3 + 6x^2) \\
 & -2x^2 - 4x \\
 & -(-2x^2 - 12x) \\
 & 8x + 48 \\
 & -(8x + 48) \\
 & 0
\end{array}
$$

hence $f(x) = (x+6)(x^2 - 2x + 8)$

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**Question 5 continued**

b)

$2\log_2(x+2) = \log_2[(x+2)^2]$ $\quad a\log b = \log(b^a)$

$= \log_2(x^2+4x+4)$

$\log_2(x^2+4x+4) + \log_2(x) - \log_2(x-6) = 3$

$\log a + \log b = \log(a \times b)$

$\log_2(x^3+4x^2+4x) - \log_2(x-6) = 3$

$\log a - \log b = \log\left(\frac{a}{b}\right)$

$\log_2\left(\frac{x^3+4x^2+4x}{x-6}\right) = 3$ $\quad \log_2 a = 3,\ a = 2^3$

$\frac{x^3+4x^2+4x}{x-6} = 8 \Rightarrow x^3+4x^2+4x = 8x-48$

$= x^3+4x^2-4x+48 = 0$

as required

**Question 5 continued**

c)

$f(x) = x^3 + 4x^2 - 4x + 48 = (x+6)(x^2 - 2x + 8) = 0$

$(x+6)(x^2 - 2x + 8) = 0$ $ab = 0$

$\hookrightarrow a = 0$

$b = 0$

$x + 6 = 0$, $x = -6$.

$\log_2(x)$ is only valid when $x > 0$. Since $-6 < 0$,

$x = -6$ is not a valid root.

$x^2 - 2x + 8 = 0$ $b^2 - 4ac > 0 \rightarrow$ 2 real roots

$ax^2 + bx + c = 0$ $b^2 - 4ac < 0 \rightarrow$ no real roots.

$b^2 - 4ac = (-2)^2 - 4(1)(8)$

$= 4 - 32$

$= -28 \rightarrow -28 < 0 \therefore$ no real roots

Since neither $(x+6) = 0$ or $x^2 - 2x + 8$ produces valid real roots, there are no real roots to the equation.

**(Total for Question 5 is 10 marks)**

**6.**

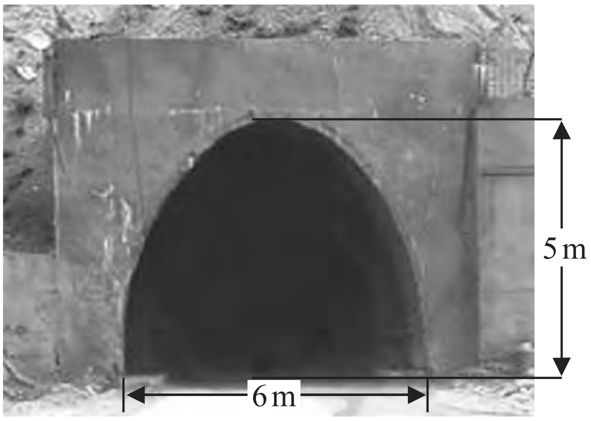
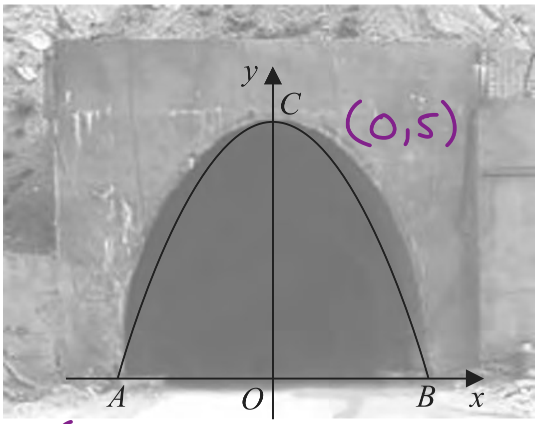



**Figure 2**




(-3,0) (3,0)

**Figure 3**

Figure 2 shows the entrance to a road tunnel. The maximum height of the tunnel is measured as 5 metres and the width of the base of the tunnel is measured as 6 metres.

Figure 3 shows a quadratic curve $BCA$ used to model this entrance.

The points $A$, $O$, $B$ and $C$ are assumed to lie in the same vertical plane and the ground $AOB$ is assumed to be horizontal.

(a) Find an equation for curve $BCA$.

**(3)**

A coach has height 4.1 m and width 2.4 m. → 1.2 m Either Side

(b) Determine whether or not it is possible for the coach to enter the tunnel.

**(2)**

(c) Suggest a reason why this model may not be suitable to determine whether or not the coach can pass through the tunnel.

**(1)**

**Question 6 continued**

a)

$y = k(x - x_1)(x - x_2)$ where $x_1$ and $x_2$ are roots.

$x_1 = -3$

$x_2 = 3$

$y = k(x+3)(x-3)$

$y = k(x^2 - 9)$ at $(0,5)$, $5 = k(0^2 - 9)$

$5 = -9k$

$k = -\frac{5}{9}$

$y = -\frac{5}{9}(x^2 - 9)$ $\therefore$ $y = \frac{5}{9}(9 - x^2)$, $\{-3 \leq x \leq 3\}$

**Question 6 continued**

b)

$y = \frac{5}{9}(9 - x^2)$

at $x = 1.2m$, $y = \frac{5}{9}(9 - (1.2)^2)$

$= 4.2m$

Since $4.2m > 4.1m$

height of tunnel > height of bus

$\therefore$ yes, bus can pass through tunnel

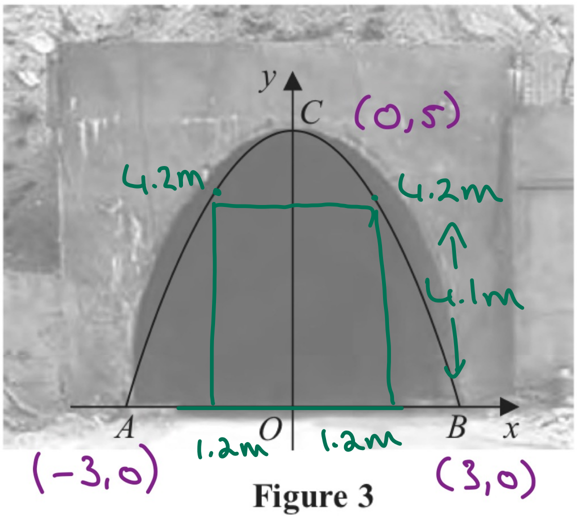



**Figure 3**

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**Question 6 continued**

c)

The quadratic curve BCA only applies to the entrance.

We do not know if the model is valid throughout the entire length of the tunnel.

Alternative Answers

| | |
|---|---|
| **(c)** | E.g.<br>• Coach needs to enter through the centre of the tunnel. This will only be possible if it is a one-way tunnel<br>• In real-life the road may be cambered (and not horizontal)<br>• The quadratic curve *BCA* is modelled for the entrance to the tunnel but we do not know if this curve is valid throughout the entire length of the tunnel<br>• There may be overhead lights in the tunnel which may block the path of the coach |

**(Total for Question 6 is 6 marks)**

**7.**

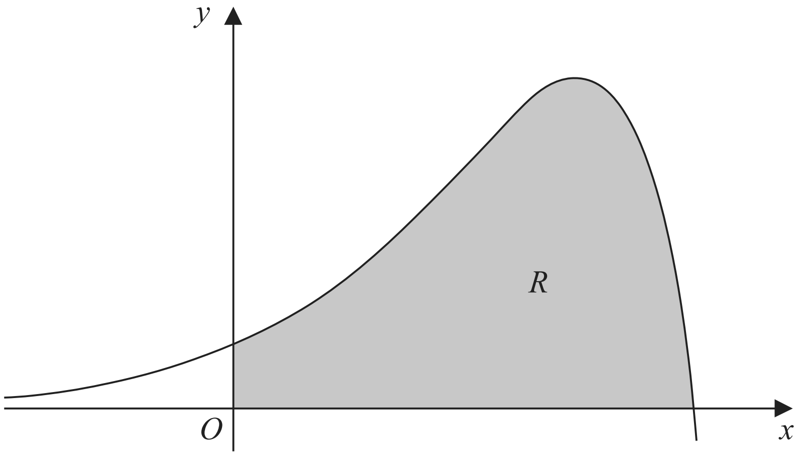


**Figure 4**

Figure 4 shows a sketch of part of the curve with equation

$$y = 2e^{2x} - xe^{2x}, \qquad x \in \mathbb{R}$$

The finite region $R$, shown shaded in Figure 4, is bounded by the curve, the $x$-axis and the $y$-axis.

Use calculus to show that the exact area of $R$ can be written in the form $pe^4 + q$, where $p$ and $q$ are rational constants to be found.
(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

**(5)**

**Question 7 continued**

a)

root: $2e^{2x} - xe^{2x} = 0$

$e^{2x}(2-x) = 0$

$e^{2x} = 0$ ↳ invalid

$2 - x = 0$

$x = 2$

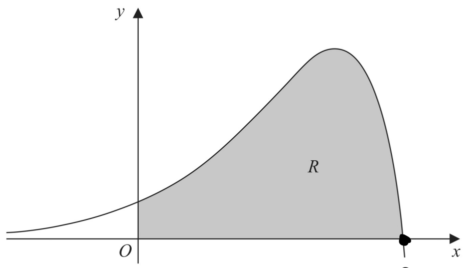


Figure 4

$R = \int_0^2 2e^{2x} - xe^{2x}\,dx$

$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$

$= \int_0^2 2e^{2x}\,dx - \int_0^2 xe^{2x}\,dx$

$\int u\,dv = uv - \int v\,du$

$= \left[ e^{2x} - \left( \frac{xe^{2x}}{2} - \int_0^2 \frac{e^{2x}}{2}\,dx \right) \right]_0^2$

$u = x \quad v = e^{2x}/2$

$du = 1 \quad dv = e^{2x}$

$= \left[ e^{2x} - \left( \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} \right) \right]_0^2$

$= \left(e^4 - \left(e^4 - \frac{e^4}{4}\right)\right) - \left(1 - \left(0 - \frac{1}{4}\right)\right)$

$= \left(\cancel{e^4} - \cancel{e^4} + \frac{e^4}{4}\right) - \left(1 - 0 + \frac{1}{4}\right)$

$= \frac{e^4}{4} - \frac{5}{4} \quad \therefore R = \frac{1}{4}e^4 - \frac{5}{4} \quad \left(p = \frac{1}{4},\ q = -\frac{5}{4}\right)$

**(Total for Question 7 is 5 marks)**

**8.** There were 2100 tonnes of wheat harvested on a farm during 2017.

The mass of wheat harvested during each subsequent year is expected to increase by 1.2% per year.

(a) Find the total mass of wheat expected to be harvested from 2017 to 2030 inclusive, giving your answer to 3 significant figures.

↳ 14 Years

**(2)**

Each year it costs

- £5.15 per tonne to harvest the first 2000 tonnes of wheat
- £6.45 per tonne to harvest wheat in excess of 2000 tonnes

(b) Use this information to find the expected cost of harvesting the wheat from 2017 to 2030 inclusive. Give your answer to the nearest £1000

**(3)**

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**Question 8 continued**

a)

2017 2018 2019

$\downarrow$ $\downarrow$ $\downarrow$

2100 T $\quad 2100 \times 1.012$ T $\quad 2100 \times 1.012^2$ T

$ar^n$, $n = 14$ (number of years).

$$S_n = \frac{a(1-r^n)}{1-r} \Rightarrow S_{14} = \frac{2100(1-1.012^{14})}{1-1.012}$$

$= 31806.99\ldots$ T

$= 31800$ T (3.s.f)

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**Question 8 continued**

b)

Total wheat after 14 years = 31806.9948... T

in 2017, 2100T

→ 14 years: first 2000T are: £5.15 × 2000 × 14
= £144,200

Amount of wheat charged @ £6.45 is:

31806.9948 − 2000 × 14 = 3806.9948

remaining wheat is : 3806.9948 × £6.45
= £24 555.11646...

Total cost = £144,200 + £24,555.11646...
= £168,755.116...
= £169,000. (nearest thousand).

**(Total for Question 8 is 5 marks)**

**9.** The curve $C$ has equation

$$y = 2x^3 + 5$$

The curve $C$ passes through the point $P(1, 7)$.

Use differentiation from first principles to find the value of the gradient of the tangent to $C$ at $P$.

**(5)**

$$f'(x) = \frac{dy}{dx} = \lim_{h \to 0}\left[\frac{f(x+h) - f(x)}{x+h-x}\right]$$

$$\frac{dy}{dx} = \lim_{h \to 0}\left(\frac{(2(x+h)^3 + 5) - (2x^3 + 5)}{h}\right)$$

$$(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$$

$$= \lim_{h \to 0}\left(\frac{\cancel{2x^3} + 6x^2h + 6xh^2 + 2h^3 \cancel{+5} \cancel{-2x^3} \cancel{-5}}{h}\right)$$

$$= \lim_{h \to 0}\left(\frac{6x^2h + 6xh^2 + 2h^3}{h}\right)$$

$$= \lim_{h \to 0}\left(6x^2 + 6xh + 2h^2\right) = 6x^2 + 0 + 0$$

$$\frac{dy}{dx} = 6x^2. \quad \therefore \left.\frac{dy}{dx}\right|_{x=1} = 6(1)^2 = 6$$



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**10.** The function f is defined by

$$\mathrm{f}: x \mapsto \frac{3x-5}{x+1}, \quad x \in \mathbb{R}, \ x \neq -1$$

(a) Find $\mathrm{f}^{-1}(x)$.

**(3)**

(b) Show that

$$\mathrm{ff}(x) = \frac{x+a}{x-1}, \quad x \in \mathbb{R}, \ x \neq \pm 1$$

where $a$ is an integer to be found.

**(4)**

The function g is defined by

$$\mathrm{g}: x \mapsto x^2 - 3x, \quad x \in \mathbb{R}, 0 \leqslant x \leqslant 5$$

(c) Find the value of $\mathrm{fg}(2)$.

**(2)**

(d) Find the range of g.

**(3)**

(e) Explain why the function g does not have an inverse.

**(1)**

**Question 10 continued**

a)

if $f(x) = \dfrac{3x-5}{x+1}$

$$x = \frac{3f^{-1}(x)-5}{f^{-1}(x)+1}$$

$$x(f^{-1}(x)+1) = 3f^{-1}(x) - 5$$

$$xf^{-1}(x) + x = 3f^{-1}(x) - 5$$

$$(x-3)f^{-1}(x) = -x-5$$

$$f^{-1}(x) = \frac{-x-5}{x-3}$$

$$\therefore f^{-1}(x) = \frac{x+5}{3-x}$$

**Question** 10 **continued**

b)

$f(x) = \frac{3x-5}{x+1}$ $\qquad ff(x) = f(f(x))$

$$ff(x) = \frac{3\left(\frac{3x-5}{x+1}\right) - 5}{\frac{3x-5}{x+1} + 1}$$

$\frac{a}{b} + c = \frac{a}{b} + \frac{bc}{b} = \frac{a+bc}{b}$

$$= \frac{3\left(\frac{3x-5}{x+1}\right) - \frac{5(x+1)}{x+1}}{\frac{3x-5}{x+1} + \frac{x+1}{x+1}} = \frac{\frac{3(3x-5)-5(x+1)}{\cancel{x+1}}}{\frac{3x-5+x+1}{\cancel{x+1}}}$$

$$= \frac{9x-15-5x-5}{3x-5+x+1}$$

$$= \frac{4x-20}{4x-4}$$

$$= \frac{x-5}{x-1}$$

$ff(x) = \frac{x-5}{x-1} \quad \therefore a = -5$

**Question 10 continued**

c)

$f(x) = \dfrac{3x-5}{x+1}$ $fg(a) = f(g(a))$

Step 1: $g(2) = 2^2 - 3(2) = 4 - 6 = -2$

Step 2: $f(g(2)) = f(-2) = \dfrac{3(-2)-5}{-2+1}$

$= \dfrac{-6-5}{-1}$

$= \dfrac{-11}{-1} = 11$

$\therefore fg(2) = 11$

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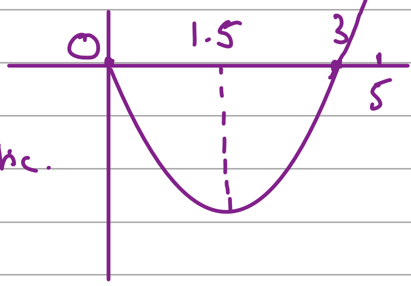
**Question 10 continued**

d)

$g(x) = x^2 - 3x \quad x \in \mathbb{R}, \; 0 \leq x \leq 5$

m.p = y. coordinate of TP

↳ inbetween roots of quadratic.



$x^2 - 3x = 0 \quad \therefore \quad x(x-3) = 0$

↳ $x = 0$

$x - 3 = 0 \;\therefore\; x = 3$

x - coordinate of tp. is $\frac{0+3}{2} = 1.5$

y. coordinate of tp. is $g(1.5) = -2.25$

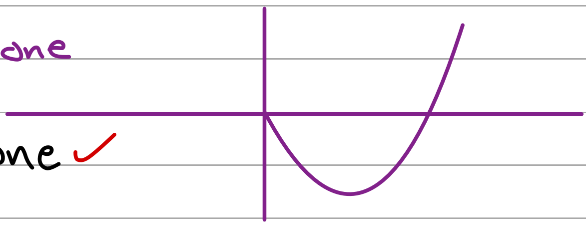
$\therefore \; g_{min} = -2.25$ ✓

$g_{max} = g(5) = 5^2 - 3 \times 5 = 25 - 15 = 10.$ ✓

$\therefore \; -2.25 \leq g(x) \leq 10$ ✓

e)

if inverse, function is one-to-one



The function g is many-to-one ✓

e.g. $g(0) = 0$ and $g(3) = 0 \;\therefore\; g(0) = g(3) = 0$

**(Total for Question 10 is 13 marks)**

**11.**

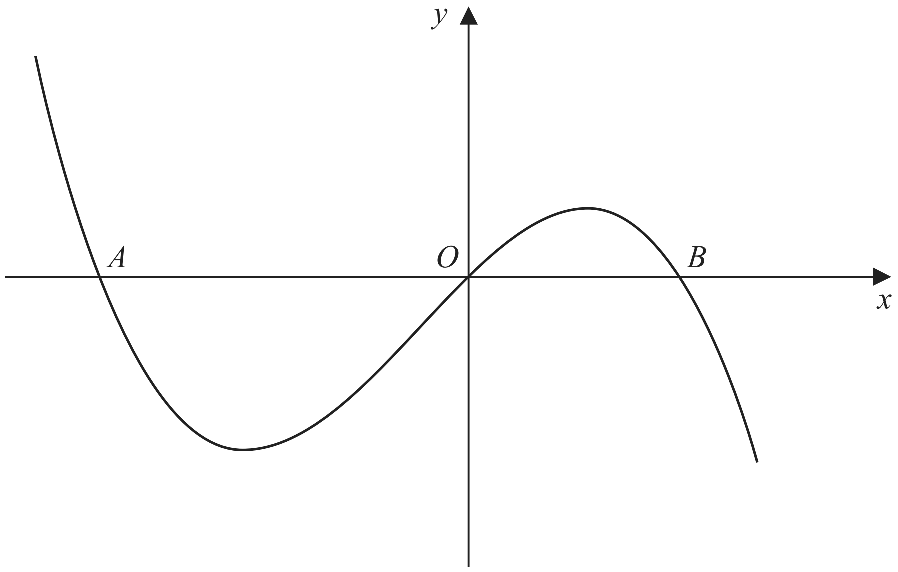



**Figure 5**

Figure 5 shows a sketch of the curve $C$ with equation $y = \mathrm{f}(x)$.
The curve $C$ crosses the $x$-axis at the origin, $O$, and at the points $A$ and $B$ as shown in Figure 5.

Given that

$$\mathrm{f}'(x) = k - 4x - 3x^2$$

where $k$ is constant,

(a) show that $C$ has a point of inflection at $x = -\frac{2}{3}$ **(3)**

Given also that the distance $AB = 4\sqrt{2}$

(b) find, showing your working, the integer value of $k$. **(7)**

**Question 11 continued**

a)

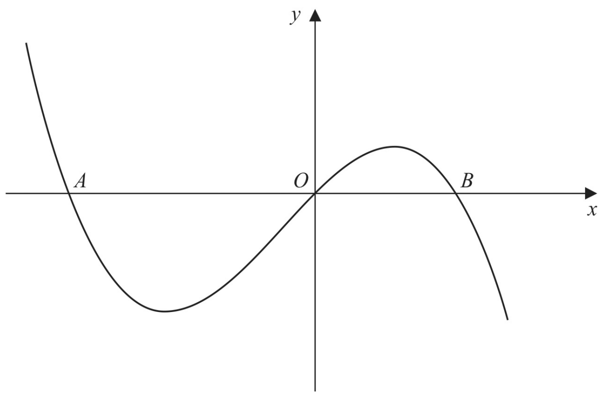



**Figure 5**

$f''(x) = 0 - 4 - 6x$

$\therefore f''(x) = -4 - 6x$

Possible POI when $f''(x) = 0$

$-4 - 6x = 0$

$6x = -4$

$x = -\frac{2}{3}$

$-\frac{2}{3} = -0.\dot{6}$ → Testing range is: $-0.7 < x < -0.6$

$f''(-0.7) = -4 - 6(-0.7) = 0.2 > 0$

$f''(-0.6) = -4 - 6(-0.6) = -0.4 < 0$

Both criteria are met & $\therefore$ C has a point of inflection at $x = -\frac{2}{3}$

**Question 11 continued**

b)

$f'(x) = k - 4x - 3x^2$

$f(x) = \int k - 4x - 3x^2 \, dx$

$= kx - 2x^2 - x^3 + c$

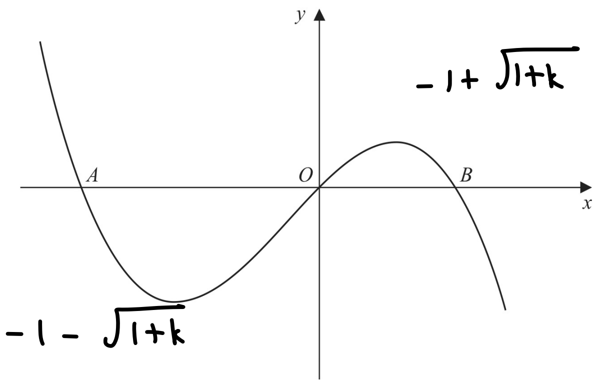



**Figure 5**

@ $x = 0$, $f(x) = 0$

$0 = k(0) - 2(0)^2 - 0^3 + c \quad \therefore c = 0$

$f(x) = kx - 2x^2 - x^3$

$= x(k - 2x - x^2)$

$\hookrightarrow x = 0$ <u>or</u> $k - 2x - x^2 = 0$

$x^2 + 2x - k = 0$

$ax^2 + bx + c, \quad x_1, x_2 = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\therefore x_1, x_2 = \dfrac{-2 \pm \sqrt{4 - (-4k)}}{2} \rightarrow 4 + 4k$

$= \dfrac{-2 \pm 2\sqrt{1+k}}{2} = -1 \pm \sqrt{1+k}$

$AB = (-1 + \sqrt{k+1}) - (-1 - \sqrt{k+1}) = 4\sqrt{2}$

$= \cancel{-1} + \sqrt{k+1} \cancel{+1} + \sqrt{k+1}$

$= 2\sqrt{k+1} = 4\sqrt{2} \Rightarrow \sqrt{k+1} = \sqrt{8}$

$k + 1 = 8 \quad \therefore k = 7.$

**(Total for Question 11 is 10 marks)**

**12.** Show that

$$\int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{1+\cos\theta}\, d\theta = 2 - 2\ln 2$$

**(7)**

Try and make denominator 1 term → Easier to solve.

$$\int \frac{\sin(2\theta)}{1+\cos\theta}\, d\theta = \int \frac{2\cancel{\sin\theta}\cos\theta}{u} \times \frac{du}{-\cancel{\sin\theta}}$$

$u = 1+\cos\theta$

$\frac{du}{d\theta} = -\sin\theta$

$$= -2\int \frac{\cos\theta}{u}\, du$$

$d\theta = \frac{du}{-\sin\theta}$

$\sin(2\theta) = 2\sin\theta\cos\theta$

$$= -2\int \frac{u-1}{u}\, du$$

at $\theta = \frac{\pi}{2}$, $u = 1$

$$= -2\int 1 - \frac{1}{u}\, du$$

$\theta = 0$, $u = 2$

$$= -2(u - \ln u)$$

Since $u = 1+\cos\theta$

$\cos\theta = u - 1$

$$\int_0^{\frac{\pi}{2}} \frac{\sin(2\theta)}{1+\cos\theta}\, d\theta = \Big[-2(u - \ln u)\Big]_2^1$$

$$= (-2(1-\ln 1)) - (-2(2-\ln 2))$$

$$= -2 + 4 - 2\ln 2$$

$$= 2 - 2\ln 2 \text{ as required.}$$

**13.** (a) Express $2\sin\theta - 1.5\cos\theta$ in the form $R\sin(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

State the value of $R$ and give the value of $\alpha$ to 4 decimal places.

**(3)**

Tom models the depth of water, $D$ metres, at Southview harbour on 18th October 2017 by the formula

$$D = 6 + 2\sin\left(\frac{4\pi t}{25}\right) - 1.5\cos\left(\frac{4\pi t}{25}\right), \qquad 0 \leqslant t \leqslant 24$$

where $t$ is the time, in hours, after 00:00 hours on 18th October 2017.

$\hookrightarrow t = 0$

Use Tom's model to

(b) find the depth of water at 00:00 hours on 18th October 2017,

**(1)**

(c) find the maximum depth of water,

**(1)**

(d) find the time, in the afternoon, when the maximum depth of water occurs. Give your answer to the nearest minute.

**(3)**

Tom's model is supported by measurements of $D$ taken at regular intervals on 18th October 2017. Jolene attempts to use a similar model in order to model the depth of water at Southview harbour on 19th October 2017.

Jolene models the depth of water, $H$ metres, at Southview harbour on 19th October 2017 by the formula

$$H = 6 + 2\sin\left(\frac{4\pi x}{25}\right) - 1.5\cos\left(\frac{4\pi x}{25}\right), \qquad 0 \leqslant x \leqslant 24$$

where $x$ is the time, in hours, after 00:00 hours on 19th October 2017.

By considering the depth of water at 00:00 hours on 19th October 2017 for both models,

$\hookrightarrow t = 24\text{Hr}$ $\qquad \hookrightarrow x = 0\text{Hr}$

(e) (i) explain why Jolene's model is not correct,

(ii) hence find a suitable model for $H$ in terms of $x$.

**(3)**

**Question 13 continued**

a)

$a\sin\theta \pm b\cos\theta = R\sin(\theta \pm \alpha)$

$R = \sqrt{a^2+b^2} \qquad \tan\alpha = \frac{b}{a}$

$a = 2$ and $b = 1.5 \Rightarrow R = \sqrt{2^2+1.5^2} = 2.5$

$\alpha = \tan^{-1}\left(\frac{1.5}{2}\right) = 0.6435$

$\therefore 2\sin\theta - 1.5\cos\theta = 2.5\sin(\theta - 0.6435)$

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**Question 13 continued**

b)

$2\sin\theta - 1.5\cos\theta = 2.5\sin(\theta - 0.6435)$

$D = 6 + 2.5\sin\left(\frac{4\pi t}{25} - 0.6435\right)$

$= 6 + 2.5\sin(-0.6435)$

$= 4.5\text{m}$ (2.s.f).

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**Question 13 continued**

c)

$D = 6 + 2.5 \sin\left(\frac{4\pi t}{25} - 0.6435\right)$

max when $\sin\left(\frac{4\pi t}{25} - 0.6435\right) = 1$

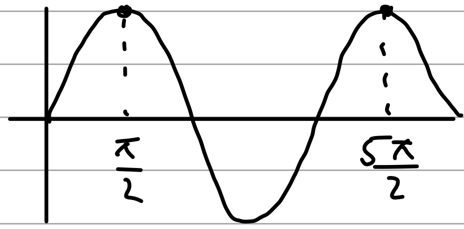
$D_{max} = 6 + 2.5(1) = 8.5\text{ m}$ ✓

**(Total for Question 1 is 6 marks)**

**Question 13 continued**

d)

$$D = 6 + 2.5\sin\left(\frac{4\pi t}{25} - 0.6435\right)$$




$$\frac{4\pi t}{25} - 0.6435 = \frac{5\pi}{2}$$

$$\frac{4\pi t}{25} = \frac{5\pi}{2} + 0.6435 \qquad \frac{ax}{b} = y, \; x = \frac{by}{a}$$

$$t = \frac{25}{4\pi}\left(\frac{5\pi}{2} + 0.6435\right)$$

$= 16.9$ Hr after 00:00 $\Rightarrow$ 16:54

**Question 13 continued**

e)

i)

$D = 6 + 2.5\sin\left(\frac{4\pi t}{25} - 0.6435\right)$

$= 6 + 2.5\sin\left(\frac{4\pi(24)}{25} - 0.6435\right) = 3.72\text{ m}$ ✓

$H = 6 + 2.5\sin\left(\frac{4\pi x}{25} - 0.6435\right)$

$= 6 + 2.5\sin\left(-0.6435\right) = 4.5\text{ m}$

Since $D = 3.72\text{m} \neq H = 4.5\text{m}$, Jolenes model is not true. ✓

ii) $H = 6 + 2\sin\left(\frac{4\pi(x+24)}{25}\right) - 1.5\cos\left(\frac{4\pi(x+24)}{25}\right)$ ✓

$0 \leqslant x < 24$

**(Total for Question 13 is 11 marks)**

**14.**

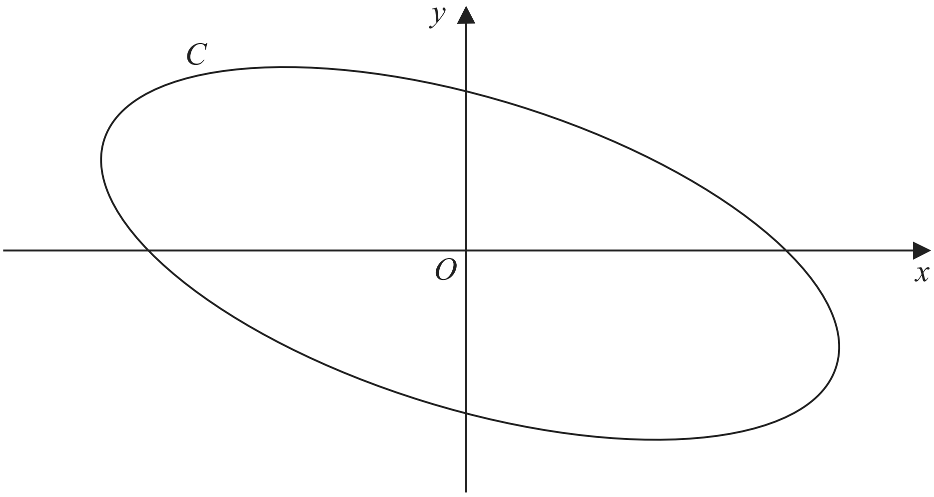



**Figure 6**

Figure 6 shows a sketch of the curve $C$ with parametric equations

$$x = 4\cos\left(t + \frac{\pi}{6}\right), \quad y = 2\sin t, \qquad 0 < t \leqslant 2\pi$$

Show that a Cartesian equation of $C$ can be written in the form

$$(x + y)^2 + ay^2 = b$$

where $a$ and $b$ are integers to be found.

**(5)**

**Question 14 continued**

$x = 4\cos\left(t + \frac{\pi}{6}\right) \quad y = 2\sin t$

$x + y = 4\cos\left(t + \frac{\pi}{6}\right) + 2\sin t$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

$x + y = 4\left(\cos t \cos\overset{\sqrt{3}/2}{\frac{\pi}{6}} - \sin t \sin\overset{1/2}{\frac{\pi}{6}}\right) + 2\sin t$ ✓✓

$x + y = 2\sqrt{3}\cos t - \cancel{2\sin t} + \cancel{2\sin t} \Rightarrow x + y = 2\sqrt{3}\cos t$ ✓

$\frac{x+y}{2\sqrt{3}} = \cos t \qquad \cos^2 t + \sin^2 t = 1$

$\cos^2 t = 1 - \sin^2 t$

$\left(\frac{x+y}{2\sqrt{3}}\right)^2 = 1 - \sin^2 t \qquad y = 2\sin t \Rightarrow \sin^2 t = \frac{y^2}{4}$

$\left(\frac{x+y}{2\sqrt{3}}\right)^2 + \frac{y^2}{4} = 1$ ✓ $\Rightarrow \frac{(x+y)^2}{12} + \frac{y^2}{4} = 1$

$(x+y)^2 + 3y^2 = 12$ ✓

$\therefore a = 3, \quad b = 12$

**(Total for Question 14 is 5 marks)**