Please check the examination details below before entering your candidate information

Candidate surname	Other names

Pearson Edexcel Level 3 GCE

Centre Number | Candidate Number

Wednesday 14 October 2020

Afternoon (Time: 2 hours) | Paper Reference **9MA0/02**

Mathematics

Advanced
Paper 2: Pure Mathematics 2

You must have:
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

Candidates may use any calculator allowed by Pearson regulations. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need*.
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 16 questions in this question paper. The total mark for this paper is 100.
- The marks for each question are shown in brackets – *use this as a guide as to how much time to spend on each question*.

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over ▶

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P 6 6 7 8 6 A 0 1 5 2

Pearson

1 The table below shows corresponding values of x and y for $y = \sqrt{\dfrac{x}{1+x}}$

The values of y are given to 4 significant figures.

x	0.5	1	1.5	2	2.5
y	0.5774	0.7071	0.7746	0.8165	0.8452

(a) Use the trapezium rule, with all the values of y in the table, to find an estimate for

$$\int_{0.5}^{2.5} \sqrt{\frac{x}{1+x}}\, dx$$

giving your answer to 3 significant figures.

(3)

a) Trapezium Rule : $\int_{x_0}^{x_n} f(x)\, dx = \frac{1}{2}\cdot h\left[(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\right]$

	x_0	x_1	x_2	x_3	x_4
x	0.5	1	1.5	2	2.5
y	0.5774	0.7071	0.7746	0.8165	0.8452
	y_0	y_1	y_2	y_3	y_4

0.5 0.5 0.5 0.5

What is h? h is the difference between each value of x.

$\Rightarrow h = \underline{0.5}$ ①

$$\Rightarrow \int_{0.5}^{2.5} \sqrt{\frac{x}{1+x}}\, dx \approx \frac{1}{2} \times 0.5\left[(0.5774 + 0.8452) + 2(0.7071 + 0.7746 + 0.8165)\right]$$ ①

$$\approx 1.50475\ldots$$

$$\Rightarrow \int_{0.5}^{2.5} \sqrt{\frac{x}{1+x}}\, dx \approx \underline{1.50}$$ ①

(b) Using your answer to part (a), deduce an estimate for $\int_{0.5}^{2.5} \sqrt{\frac{9x}{1+x}}\,dx$

(1)

b) From part a : $\int_{0.5}^{2.5} \sqrt{\frac{x}{1+x}}\,dx \approx 1.50$

this is a constant, so we can take it out the integral.

$$\int_{0.5}^{2.5} \sqrt{\frac{9x}{1+x}}\,dx = \int_{0.5}^{2.5} \sqrt{9}\sqrt{\frac{x}{1+x}}\,dx = 3\underbrace{\int_{0.5}^{2.5} \sqrt{\frac{x}{1+x}}\,dx}_{\text{this is the same as we had in part a}}$$

$$\Rightarrow \int_{0.5}^{2.5} \sqrt{\frac{9x}{1+x}}\,dx \approx 3 \times 1.50 = \underline{4.50} \quad 1$$

Given that

$$\int_{0.5}^{2.5} \sqrt{\frac{9x}{1+x}}\,dx = 4.535 \text{ to 4 significant figures}$$

(c) comment on the accuracy of your answer to part (b).

(1)

c) Estimate from part b : $\int_{0}^{2.5} \sqrt{\frac{9x}{1+x}}\,dx \approx 4.50$

The accuracy of the answer in part b is high, since $\underset{\text{estimate}}{4.50} \approx \underset{\text{real value}}{4.535}$ 1

2. Relative to a fixed origin, points P, Q and R have position vectors $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ respectively.

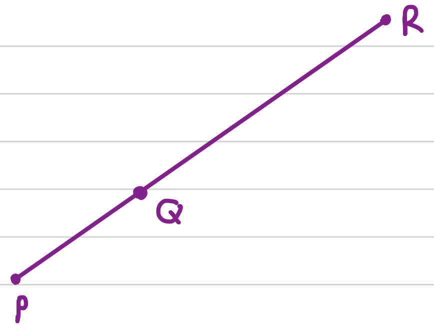
Given that

- P, Q and R lie on a straight line
- Q lies one third of the way from P to R

show that

$$\mathbf{q} = \frac{1}{3}(\mathbf{r} + 2\mathbf{p})$$

(3)



$\overrightarrow{QR} = \frac{2}{3}\overrightarrow{PR}$

$\underline{r} - \underline{q} = \frac{2}{3}(\underline{r} - \underline{p})$ 1

$\Rightarrow \underline{r} - \underline{q} = \frac{2}{3}\underline{r} - \frac{2}{3}\underline{p}$

$\Rightarrow \underline{q} = \underline{r} - \frac{2}{3}\underline{r} + \frac{2}{3}\underline{p}$ 1

$\Rightarrow \underline{q} = \frac{1}{3}\underline{r} + \frac{2}{3}\underline{p}$

$\Rightarrow \underline{q} = \frac{1}{3}(\underline{r} + 2\underline{p})$ as required 1

3. (a) Given that

$$2\log(4-x) = \log(x+8)$$

show that

$$x^2 - 9x + 8 = 0$$

(3)

a) $2\log(4-x) = \log(x+8)$

log laws:

$a \cdot \log b = \log(b)^a$

and

if $\log(a) = \log(b)$ then $a = b$

$\Rightarrow \log(4-x)^2 = \log(x+8)$ 1

$\Rightarrow (4-x)^2 = x+8$ 1

$\Rightarrow 16 - 8x + x^2 = x + 8$

$\Rightarrow x^2 - 9x + 8 = 0$ as required 1

(b) (i) Write down the roots of the equation

$$x^2 - 9x + 8 = 0$$

(ii) State which of the roots in (b)(i) is **not** a solution of

$$2\log(4-x) = \log(x+8)$$

giving a reason for your answer.

(2)

b i) $x^2 - 9x + 8 = 0$

$\Rightarrow (x-8)(x-1) = 0$

$\Rightarrow x = 1$ and $x = 8$ 1

M: 8, A: -9

$-8, -1$

these are our roots.

log(a) is only valid for $a > 0$

b ii) For $x = 8$, $2\log(4-x) = 2\log(4-8) = 2\log(-4)$; hence $x = 8$ is not valid since $2\log(-4)$ cannot be found. 1

4. In the binomial expansion of

$$(a + 2x)^7 \qquad \text{where } a \text{ is a constant}$$

the coefficient of x^4 is 15 120

Find the value of a.

(3)

Formula : k^{th} term of $(x+y)^n = \binom{n}{k} \cdot x^k y^{n-k}$

$\Rightarrow \binom{7}{4} (2x)^4 a^3$ ① $\Rightarrow \binom{7}{4} \cdot 2^4 \cdot a^3 = 15120$

$\Rightarrow 560a^3 = 15120$ ①

$\Rightarrow a = \sqrt[3]{\frac{15120}{560}} = 3 \quad \Rightarrow \underline{a = 3}$ ①

5. The curve with equation $y = 3 \times 2^x$ meets the curve with equation $y = 15 - 2^{x+1}$ at the point P.

Find, using algebra, the exact x coordinate of P.

(4)

$y = 3 \cdot 2^x$ and $y = 15 - 2^{x+1}$ Find point of Intersection!

$\Rightarrow \quad 3 \cdot 2^x = 15 - 2^{x+1}$ 1 $\quad * \; 2^{x+1} = 2^x \cdot 2^1 = 2^x \cdot 2$

$\Rightarrow \quad 3 \cdot 2^x = 15 - 2^x \cdot 2$

$\div 2^x$ on both sides

$\Rightarrow \quad 3 = \dfrac{15}{2^x} - 2$

$\Rightarrow \quad 5 = \dfrac{15}{2^x} \quad \Rightarrow \quad 2^x = \dfrac{15}{5}$

log laws: $\ln(a^b) = b\ln(a)$

'ln' of both sides

$\Rightarrow \quad 2^x = 3$ 1

$\Rightarrow \quad \ln(2^x) = \ln(3)$

$\Rightarrow \quad x \cdot \ln(2) = \ln(3)$ $\quad \div \ln(2)$

$\Rightarrow \quad x = \dfrac{\ln(3)}{\ln(2)}$

2: 1 mark for working, 1 mark for correct answer

x coordinate of P.

6. (a) Given that

$$\frac{x^2+8x-3}{x+2} \equiv Ax+B+\frac{C}{x+2} \qquad x \in \mathbb{R} \;\; x \neq -2$$

find the values of the constants A, B and C

(3)

a) Partial Fractions:

$$\frac{x^2+8x-3}{x+2} \equiv Ax+B+\frac{C}{x+2}$$

$$x^2+8x-3 \equiv Ax(x+2)+B(x+2)+C$$

let $x=-2$, then $(-2)^2+8(-2)-3 = A(-2)(-2+2)+B(-2+2)+C$ ①

$\Rightarrow \quad -15 = C$

let $x=0$, then $-3 = 2B-15 \quad \Rightarrow 12 = 2B$

$\Rightarrow \quad B = 6$

let $x=1$, then $6 = 3A + 6(3) - 15$

$\Rightarrow 6 = 3A+3$

$\Rightarrow 3 = 3A \quad \Rightarrow A = 1$

$\Rightarrow \quad A=1, \; B=6$ and $C=-15$ ②

1 mark for two correct

1 mark for all three correct

(b) Hence, using algebraic integration, find the exact value of

$$\int_0^6 \frac{x^2 + 8x - 3}{x + 2}\,dx$$

giving your answer in the form $a + b\ln 2$ where a and b are integers to be found.

(4)

b) From part a : $\displaystyle\int_0^6 \frac{x^2+8x-3}{x+2}\,dx = \int_0^6 x + 6 - \frac{15}{x+2}\,dx$

$\displaystyle\int x\,dx = \frac{x^2}{2}$

$\displaystyle\int 6\,dx = 6x$

$\displaystyle\int \frac{15}{x+2}\,dx = 15\ln(x+2)$

$= \left[\dfrac{x^2}{2} + 6x - 15\ln(x+2)\right]_0^6$ ① integrating $\frac{15}{x+2}$ ① Correct integration

$= \left(\dfrac{6^2}{2} + 6(6) - 15\ln(8)\right) - \left(-15\ln(2)\right)$

$\ln(8) = \ln(2^3) = 3\cdot\ln(2)$

$\hookrightarrow$ power log law

$= 18 + 36 - 15\ln(8) + 15\ln(2)$ ①

$= 54 - 15(3\ln(2)) + 15\ln(2)$

$= 54 - 30\ln(2)$ ①

$a + b\ln(2) \Rightarrow a = 54,$

$b = -30$

7.

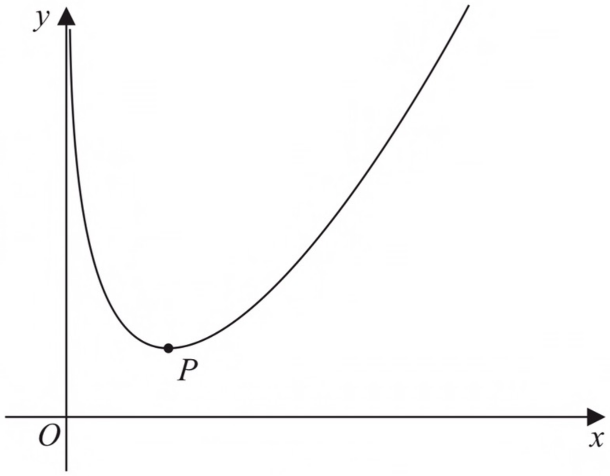


Figure 1

Figure 1 shows a sketch of the curve C with equation

$$y = \frac{4x^2 + x}{2\sqrt{x}} - 4\ln x \qquad x > 0$$

(a) Show that

$$\frac{dy}{dx} = \frac{12x^2 + x - 16\sqrt{x}}{4x\sqrt{x}}$$

(4)

a) $y = \underbrace{\frac{4x^2+x}{2\sqrt{x}}}_{} - \underbrace{4\ln x}_{}$, Find $\frac{dy}{dx}$

• Log Differentiation : $\frac{d}{dx}(\ln x) = \frac{1}{x}$

• Quotient Rule : If $h(x) = \frac{f(x)}{g(x)}$

then $h'(x) = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{(g(x))^2}$

• $\frac{d}{dx}(4\ln x) = 4 \cdot \frac{1}{x} = \boxed{\frac{4}{x}}$ 1

let $h(x) = \frac{4x^2+x}{2\sqrt{x}} \Rightarrow f(x) = 4x^2 + x \rightarrow f'(x) = 8x + 1$

$2\sqrt{x} = 2x^{1/2}$

$g(x) = 2\sqrt{x} \rightarrow g'(x) = \frac{1}{\sqrt{x}}$ 1

$\Rightarrow h'(x) = \frac{(8x+1)(2\sqrt{x}) - (4x^2+x)\left(\frac{1}{\sqrt{x}}\right)}{(2\sqrt{x})^2}$ 1 $= \frac{16x^{3/2} + 2x^{1/2} - \frac{4x^2}{x^{1/2}} - \frac{x}{x^{1/2}}}{4x} = \frac{16x^{3/2} + 2x^{1/2} - 4x^{3/2} - x^{1/2}}{4x}$

$\Rightarrow h'(x) = \frac{12x^{3/2} + x^{1/2}}{4x} = 3x^{1/2} + \frac{1}{4x^{1/2}} = \boxed{3\sqrt{x} + \frac{1}{4\sqrt{x}}}$

$\Rightarrow \frac{dy}{dx} = 3\sqrt{x} + \frac{1}{4\sqrt{x}} - \frac{4}{x} = \frac{12x+1}{4\sqrt{x}} - \frac{4}{x} = \frac{12x^2 + x - 16\sqrt{x}}{4x\sqrt{x}} = \frac{dy}{dx}$ as required. 1

The point P, shown in Figure 1, is the minimum turning point on C.

(b) Show that the x coordinate of P is a solution of

$$x = \left(\frac{4}{3} - \frac{\sqrt{x}}{12}\right)^{\frac{2}{3}}$$

(3)

b) From part a : $\frac{dy}{dx} = \frac{12x^2 + x - 16\sqrt{x}}{4x\sqrt{x}}$

Our first step is to set $\frac{dy}{dx} = 0.$ $\Rightarrow$ $\frac{12x^2 + x - 16\sqrt{x}}{4x\sqrt{x}} = 0$

$\Rightarrow$ $12x^2 + x - 16\sqrt{x} = 0$ $\div \sqrt{x}$

$\Rightarrow$ $12x^{3/2} + \sqrt{x} - 16 = 0$ (1)

$\Rightarrow$ $12x^{3/2} = 16 - \sqrt{x}$ $\div 12$ (1)

$\Rightarrow$ $x^{3/2} = \frac{16}{12} - \frac{\sqrt{x}}{12}$

$\Rightarrow$ $x^{3/2} = \frac{4}{3} - \frac{\sqrt{x}}{12}$

$\Rightarrow$ $x = \left(\frac{4}{3} - \frac{\sqrt{x}}{12}\right)^{2/3}$ as required. (1)

(c) Use the iteration formula

$$x_{n+1} = \left(\frac{4}{3} - \frac{\sqrt{x_n}}{12}\right)^{\frac{2}{3}} \qquad \text{with } x_1 = 2$$

to find (i) the value of x_2 to 5 decimal places,

(ii) the x coordinate of P to 5 decimal places.

(3)

c)

i) $x_1 = 2$ and $x_{n+1} = \left(\frac{4}{3} - \frac{\sqrt{x_n}}{12}\right)^{2/3}$ $\Rightarrow$ $x_2 = \left(\frac{4}{3} - \frac{\sqrt{x_1}}{12}\right)^{2/3} = \left(\frac{4}{3} - \frac{\sqrt{2}}{12}\right)^{2/3}$ (1)

Sub this in!

$x_2 = 1.138935...$

$x_2 = 1.13894$ (5 d.p) (1)

ii) $x = 1.15650$ (1)

8. A curve C has equation $y = \mathrm{f}(x)$

Given that

- $\mathrm{f}'(x) = 6x^2 + ax - 23$ where a is a constant
- the y intercept of C is -12
- $(x + 4)$ is a factor of $\mathrm{f}(x)$

find, in simplest form, $\mathrm{f}(x)$

(6)

$f'(x) = 6x^2 + ax - 23$

Integration: $\int 6x^2\,dx = \frac{6x^3}{3} = 2x^3$

$\int ax\,dx = \frac{ax^2}{2}$

$\int -23\,dx = -23x$

$\Rightarrow f(x) = \int f'(x)\,dx$ ①

$\Rightarrow f(x) = \int 6x^2 + ax - 23\,dx$

Constant of integration.

$f(x) = 2x^3 + \frac{ax^2}{2} - 23x + C$ ①

Y intercept (Y when x = 0)

$\Rightarrow x = 0,\ f(0) = 2(0)^3 + \frac{a \cdot 0^2}{2} - 23(0) + C = -12$

$\Rightarrow C = -12$ ①

$\Rightarrow f(x) = 2x^3 + \frac{ax^2}{2} - 23x - 12$

$\Rightarrow$ If $x+4$ is a factor of $f(x)$ then $f(-4) = 0$

$\Rightarrow f(-4) = 0 = 2(-4)^3 + \frac{a(-4)^2}{2} - 23(-4) - 12$ ①

$\Rightarrow 0 = -128 + 8a + 80$

$\Rightarrow 8a = 48 \Rightarrow a = \frac{48}{8} = \underline{6}$ ① $\Rightarrow f(x) = \underline{2x^3 + 3x^2 - 23x - 12}$ ①

9. A quantity of ethanol was heated until it reached boiling point.

The temperature of the ethanol, θ°C, at time t seconds after heating began, is modelled by the equation

$$\theta = A - Be^{-0.07t}$$

where A and B are positive constants.

Given that

- the initial temperature of the ethanol was 18 °C
- after 10 seconds the temperature of the ethanol was 44 °C

(a) find a complete equation for the model, giving the values of A and B to 3 significant figures.

(4)

a) $\theta = A - Be^{-0.07t}$

$t = 0,\ \theta = 18 \Rightarrow 18 = A - Be^{-0.07 \cdot 0}$ $\quad e^0 = 1$

$18 = A - B$ 1

$t = 10,\ \theta = 44 \Rightarrow 44 = A - Be^{-0.7}$ 1

$A = \underline{B + 18}$ $\quad A = \underline{Be^{-0.7} + 44}$ $\Rightarrow B + 18 = Be^{-0.7} + 44$

$\Rightarrow B - Be^{-0.7} = 26$

$\Rightarrow B(1 - e^{-0.7}) = 26$

$B = 51.6$ $\quad \Rightarrow B = \dfrac{26}{1 - e^{-0.7}} = 51.647\ldots \Rightarrow B = \underline{51.6}$ 1

$\Rightarrow A = 51.6 + 18 = \underline{69.6} = A$

$\Rightarrow \theta = \underline{69.6 - 51.6e^{-0.07t}}$ 1

Ethanol has a boiling point of approximately 78 °C

(b) Use this information to evaluate the model.

(2)

b) $\theta = 69.6 - 51.6e^{-0.07t}$

The maximum temperature, according to the model, is 69.6 °C. ①

=> The model is not appropriate since 69.6 °C is much lower than 78 °C. ①

10.

(a) Show that

$$\cos 3A \equiv 4\cos^3 A - 3\cos A$$

(4)

$\cos 3A = \cos(\underset{2A}{2A} + \underset{A}{A})$

Compound Angle Formula:

- $\cos(x+y) = \cos x \cos y - \sin x \sin y$

We can use the compound angle formula with $x = 2A$ and $y = A$.

Double Angle Formula:

- $\cos 2A = 2\cos^2 A - 1$
- $\sin 2A = 2\sin A \cos A$
- $\sin^2 x + \cos^2 x = 1.$

$\Rightarrow 2\sin^2 x = 2 - 2\cos^2 x$

$\Rightarrow \cos 3A = \cos 2A \cos A - \sin 2A \sin A$ ①

$= (2\cos^2 A - 1)\cos A - (2\sin A \cos A)\sin A$ ①

$= 2\cos^3 A - \cos A - 2\sin^2 A \cos A$

$= 2\cos^3 A - \cos A - (2 - 2\cos^2 A)\cos A$ ①

$= 2\cos^3 A - \cos A - 2\cos A + 2\cos^3 A$

$\Rightarrow \cos 3A \equiv 4\cos^3 A - 3\cos A$ ① as required.

(b) Hence solve, for $-90° \leqslant x \leqslant 180°$, the equation

$$1 - \cos 3x = \sin^2 x$$

(4)

$1 - \cos 3x = \sin^2 x$

- $\sin^2 x + \cos^2 x = 1$
- Part a: $\cos 3A \equiv 4\cos^3 A - 3\cos A$

$\Rightarrow 1 - \cos 3x = 1 - \cos^2 x$

let $\cos x = y$

$\Rightarrow -4y + y + 3$

$-(4y - y - 3)$

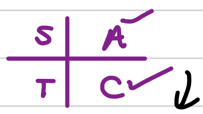
$-(4y+3)(y-1)$

$\Rightarrow 1 - (4\cos^3 x - 3\cos x) = 1 - \cos^2 x$

$\Rightarrow 1 - 4\cos^3 x + 3\cos x - 1 + \cos^2 x = 0 \Rightarrow \cos^2 x + 3\cos x - 4\cos^3 x = 0$ ①

$\Rightarrow \cos x(\cos x + 3 - 4\cos^2 x) = 0$

$\Rightarrow \cos x(4\cos x + 3)(\cos x - 1) = 0$



$\Rightarrow \cos x = 0$

$x = 90°$

$x = -90$ ①

$4\cos x + 3 = 0$

$\cos x = -\frac{3}{4} \Rightarrow x = 139°$

$\cos x - 1 = 0$ ①

$x = 0$

$\Rightarrow$ Solutions are: $x = -90°, 0°, 90°$ and $139°$. ①

11.

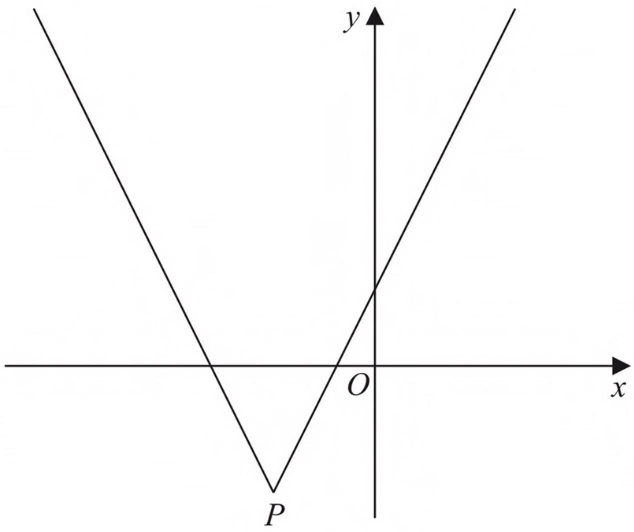


Figure 2

Figure 2 shows a sketch of the graph with equation

$$y = 2|x + 4| - 5$$

The vertex of the graph is at the point P, shown in Figure 2.

(a) Find the coordinates of P.

(2)

P is our turning point, so we can read the turning point from the equation.

$x = -4$ and $y = -5$ $\Rightarrow$ $P(-4, -5)$

1 2

(b) Solve the equation

$$3x + 40 = 2|x + 4| - 5$$

(2)

Option 1 : $3x + 40 = 2(x+4) - 5$

$3x + 40 = 2x + 3$ $\Rightarrow$ $x = -37$ $\Rightarrow$ $3(-37) + 40 = 2|-37+4| - 5$

$-71 \neq 61$

$\Rightarrow$ $x = -37$ is not a valid solution.

Option 2 : $3x + 40 = -2(x+4) - 5$

$3x + 40 = -2x - 8 - 5$

$5x = -53$

$x = -10.6$ 1 $\Rightarrow$ Check Solution : $\frac{41}{5} = \frac{41}{5}$ $\Rightarrow$ The solution is $x = -10.6$.

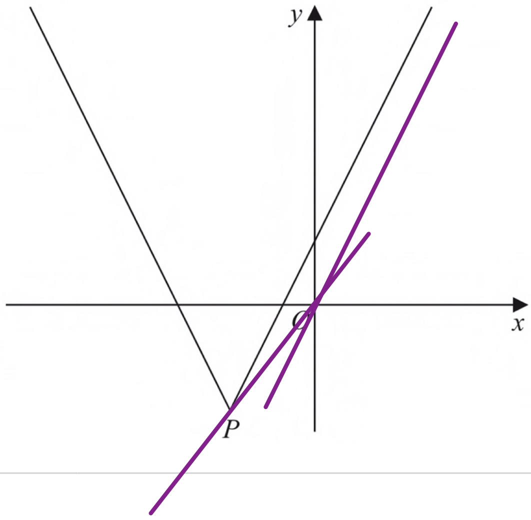
1

A line l has equation $y = ax$, where a is a constant.

Given that l intersects $y = 2|x+4| - 5$ at least once,

(c) find the range of possible values of a, writing your answer in set notation.

(3)



$y = ax$

For $a = 2$, there will never be intersection.

$\Rightarrow$ For $a > 2$, there will always be at least one point of intersection. 1

$y = ax$ with point $P(-4, -5)$

$\Rightarrow$ $-5 = -4a$ $\Rightarrow$ $a = 1.25$

$\Rightarrow$ For $a \leq 1.25$, there will be at least one point of intersection. 1

$\Rightarrow$ Set Notation : $(-\infty, 1.25] \cup (2, \infty)$ 1

12.

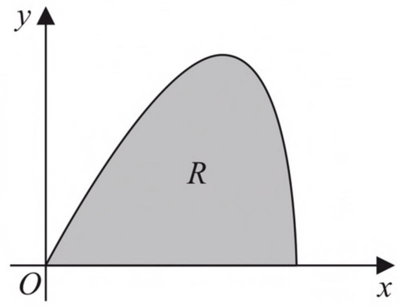


Figure 3

The curve shown in Figure 3 has parametric equations

$$x = 6\sin t \qquad y = 5\sin 2t \qquad 0 \leqslant t \leqslant \frac{\pi}{2}$$

The region R, shown shaded in Figure 3, is bounded by the curve and the x-axis.

(a) (i) Show that the area of R is given by $\int_0^{\frac{\pi}{2}} 60\sin t\cos^2 t\,\mathrm{d}t$

(3)

$R = \int_{x_1}^{x_2} y(x)\ dx = \int_{t_1}^{t_2} y(t)\ \frac{dx}{dt}\ dt$

$y(t) = y = 5\sin(2t)$

$x = 6\sin t \Rightarrow \frac{dx}{dt} = 6\cos t$

$= \int_{t_1}^{t_2} 6\cos t \cdot 5\sin 2t\ dt$ (1)

• $\sin 2t = 2\sin t\cos t$

$= \int_{t_1}^{t_2} 30\cos t \cdot \sin 2t$

$= \int_{t_1}^{t_2} 30\cos t \cdot 2\sin t\cos t\ dt = \int_{t_1}^{t_2} 60\cos^2 t\sin t\ dt$ (1)

$\Rightarrow t_1 = 0$ and $t_2 = \frac{\pi}{2} \Rightarrow R = \int_0^{\pi/2} 60\sin t\cos^2 t\ dt$ as required. (1)

(ii) Hence show, by algebraic integration, that the area of R is exactly 20

(3)

From part i: $R = \int_0^{\pi/2} 60 \cdot \sin t \cdot \cos^2 t \, dt$ $\qquad \cos^2 t = 1 - \sin^2 t$

$= 60 \cdot \int_0^{\pi/2} \sin t (1 - \sin^2 t) \, dt$

$\int \sin t \, dt = -\cos t + C$

$\int -\sin^3 t \, dt = \cos t - \frac{1}{3}\cos^3(t) + C$

$= 60 \int_0^{\pi/2} \sin t - \sin^3 t \, dt$ ①

$= 60 \left[-\frac{1}{3}\cos^3 t \right]_0^{\pi/2}$ ① $= 60 \left[-\frac{1}{3}\cos^3\left(\frac{\pi}{2}\right) - -\frac{1}{3}\cos^3(0) \right]$

$\Rightarrow R = 60\left(0 - -\frac{1}{3}\right)$

$\Rightarrow R = \underline{20}$ as required. ①

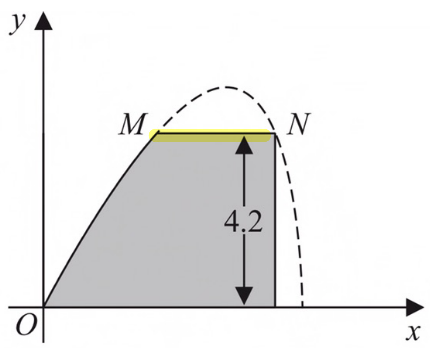


Figure 4

Part of the curve is used to model the profile of a small dam, shown shaded in Figure 4. Using the model and given that

- x and y are in metres
- the vertical wall of the dam is 4.2 metres high
- there is a horizontal walkway of width MN along the top of the dam

S | A
T | C

(b) calculate the width of the walkway.

(5)

$x = 6\sin t$, $y = 5\sin 2t \Rightarrow$ when $y = 4.2 \Rightarrow 4.2 = 5\sin 2t$

$\Rightarrow \sin 2t = \frac{4.2}{5}$ ① $\Rightarrow 2t = \sin^{-1}\left(\frac{4.2}{5}\right) \Rightarrow \underline{t_1 = 0.49865...}$

and $t_2 = \pi - 2 \times t_1 = \pi - 2 \times 0.49865 = \underline{1.0721...}$ ①

$\Rightarrow x_1 = 6\sin(0.49865) = \underline{2.869}$ and $x_2 = 6\sin(1.0721) = \underline{5.269}$ ① ①

Width of the path is going to be $x_2 - x_1 = 5.269 - 2.869$

$= \underline{2.40\text{m}}$ ①

13. The function g is defined by

$$g(x) = \frac{3\ln(x) - 7}{\ln(x) - 2} \qquad x > 0 \qquad x \neq k$$

where k is a constant.

(a) Deduce the value of k.

(1)

• When we have a fraction, the denominator cannot equal 0,

$\Rightarrow \quad \ln x - 2 = 0$

$\ln x = 2$

$e^{\ln x} = e^2$

$x = e^2 \qquad \Rightarrow \quad k = e^2 \quad$ or $\quad x \neq e^2$ ①

(b) Prove that

$$g'(x) > 0$$

for all values of x in the domain of g.

(3)

Quotient Rule If $g(x) = \frac{f(x)}{h(x)}$ then $g'(x) = \frac{f'(x)h(x) - f(x)h'(x)}{(h(x))^2}$

Recall that $g(x) = \frac{3\ln x - 7}{\ln x - 2} \Rightarrow$ let $f(x) = 3\ln x - 7$ then $f'(x) = \frac{3}{x}$ ①

$h(x) = \ln x - 2$ then $h'(x) = \frac{1}{x}$

$$\Rightarrow g'(x) = \frac{\frac{3}{x}(\ln x - 2) - \frac{1}{x}(3\ln x - 7)}{(\ln x - 2)^2} = \frac{\frac{3}{x}\cdot\ln x - \frac{6}{x} - \frac{3\ln x}{x} + \frac{7}{x}}{(\ln x - 2)^2} = \frac{1/x}{(\ln x - 2)^2}$$

$\Rightarrow g'(x) = \frac{1}{x(\ln x - 2)^2}$ ①

• We know that $x > 0$

• $(\ln x - 2)$ is squared

$\Rightarrow$ the denominator is always positive, hence $g'(x) > 0$. ①

(c) Find the range of values of a for which

$$g(a) > 0$$

(2)

Recall that $g(x) = \dfrac{3\ln x - 7}{\ln x - 2}$

let $\ln x = y$, then $g(x) = \dfrac{3y-7}{y-2} > 0$.

• Multiply both sides by $(y-2)$ $\Rightarrow$ $3y - 7 > 0$. ①

$\Rightarrow$ $y > 7/3$

$\Rightarrow$ $\ln x > 7/3$ (we change x to a now)

$\Rightarrow$ $a > e^{7/3}$

• $y = 2$ then $g(x)$ not defined since denominator equal to 0.

$\Rightarrow$ $y < 2$

$\Rightarrow$ $\ln(a) < 2$

$\Rightarrow$ $a < e^2$ $\Rightarrow$ $0 < a < e^2$ and $a > e^{7/3}$. ①

14. A circle C with radius r

- lies only in the 1st quadrant
- touches the x-axis and touches the y-axis

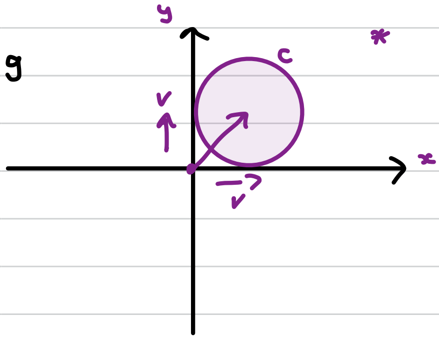
The line l has equation $2x + y = 12$

(a) Show that the x coordinates of the points of intersection of l with C satisfy

$$5x^2 + (2r - 48)x + (r^2 - 24r + 144) = 0$$

(3)

• The centre of the circle is shifted by r units along the x and the y axis.



$2x + y = 12$

$\Rightarrow$ $C: \ (x-r)^2 + (y-r)^2 = r^2$ $\qquad l: \ y = 12 - 2x$

$\Rightarrow$ $x^2 - 2rx + r^2 + y^2 - 2ry + r^2 = r^2$

$\Rightarrow$ $x^2 + y^2 - 2rx - 2ry + r^2 = 0$ 1 Substitute this in!

$\Rightarrow$ $x^2 + (12-2x)^2 - 2rx - 2r(12-2x) + r^2 = 0$

1

$\Rightarrow$ $x^2 + 144 - 48x + 4x^2 - 2rx - 24r + 4rx + r^2 = 0$

$\Rightarrow$ $5x^2 - 48x + 2rx + (r^2 - 24r + 144) = 0$

$\Rightarrow$ $5x^2 + (2r-48)x + (r^2 - 24r + 144) = 0$ as required. 1

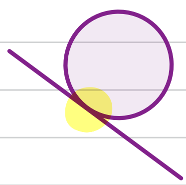
Given also that l is a tangent to C,

(b) find the two possible values of r, giving your answers as fully simplified surds.

(4)

discriminant

Tangent $\Rightarrow$ $b^2 - 4ac = 0$ Since one repeated root.



Recall from part a we have that $5x^2 + (2r-48)x + (r^2 - 24r + 144) = 0$

$a = 5, \ b = 2r - 48$ and $c = r^2 - 24r + 144$

$\Rightarrow$ $b^2 - 4ac = 0$ $\Rightarrow$ $(2r-48)^2 - 4 \times 5(r^2 - 24r + 144) = 0$ 1

$\Rightarrow$ $4r^2 - 192r + 2304 - 20r^2 + 480r - 2880 = 0$

$\Rightarrow$ $-16r^2 + 288r - 576 = 0$

$\div -16$ $\Rightarrow$ $r^2 - 18r + 36 = 0$ 1 $\Rightarrow$ Quadratic formula: $a = 1 \quad b = -18 \quad c = 36$

$\Rightarrow$ $r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{18 \pm \sqrt{(-18)^2 - 4(1)(36)}}{2}$ 1 $= \dfrac{18 \pm 6\sqrt{5}}{2}$ $\Rightarrow$ $r = \underline{9 \pm 3\sqrt{5}}$ 1

15. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

A geometric series has common ratio r and first term a.

Given $r \neq 1$ and $a \neq 0$

(a) prove that

$$S_n = \frac{a(1-r^n)}{1-r}$$

(4)

$\times r$ $S_n = a + ar + ar^2 + \dots + ar^{n-1}$ $\times r$ 1

$\Rightarrow r \cdot S_n = ar + ar^2 + ar^3 + \dots + ar^n$ 1

$\Rightarrow S_n - r \cdot S_n = a + ar + ar^2 + \dots + ar^{n-1} - ar - ar^2 - \dots - ar^n$

$\Rightarrow S_n - r \cdot S_n = a - ar^n$ 1 (we now want to rearrange and manipulate this to get the required answer / proof)

$\Rightarrow S_n(1-r) = a(1-r^n)$ $\Rightarrow$ $S_n = \dfrac{a(1-r^n)}{1-r}$ as required. 1

$\div (1-r)$

Given also that S_{10} is four times S_5

(b) find the exact value of r.

(4)

Recall that : $S_n = \dfrac{a(1-r^n)}{1-r}$

$\Rightarrow S_{10} = 4 \times S_5$

$\Rightarrow \dfrac{a(1-r^{10})}{1-r} = \dfrac{4a(1-r^5)}{1-r}$ $\div a$ $\times 1-r$ 1

$\Rightarrow 1 - r^{10} = 4(1-r^5)$ 1 $\Rightarrow 1 - r^{10} = 4 - 4r^5$

$\Rightarrow r^{10} - 4r^5 + 3 = 0$ then let $x = r^5$ and $x^2 = r^{10}$

$\Rightarrow x^2 - 4x + 3 = 0$

$\Rightarrow (x-3)(x-1) = 0$

$\Rightarrow (r^5 - 3)(r^5 - 1) = 0$ $\rightarrow$ $r^5 = 1 \Rightarrow r = 1$ (but this solution isn't valid since $r \neq 1$).

$\Rightarrow r^5 = 3$ 1

$\Rightarrow r = \sqrt[5]{3}$

$\Rightarrow$ The exact value of r is $r = \sqrt[5]{3}$ 1

16. Use algebra to prove that the square of any natural number is **either** a multiple of 3 **or** one more than a multiple of 3

(4)

• $3k$ • $3k+1$ • $3k+2$ (we can express natural in this form)

$3k$: $(3k)^2 = 9k^2 = 3 \times 3k^2$ which is a multiple of 3.

$3k+1$: $(3k+1)^2 = (3k+1)(3k+1) = 9k^2+6k+1$ (1)

$= 3(3k^2+2k)+1$ which is one more than a multiple of 3. (1)

$3k+2$: $(3k+2)^2 = (3k+2)(3k+2) = 9k^2+12k+4$ (1)

$= 3(3k^2+4k+1)+1$ which is also one more than a multiple of 3.

=) We have shown that the square of any natural number is either a multiple of 3 or one more than a multiple of 3. (1)