Pearson Edexcel Level 3

GCE Mathematics

Advanced

Paper 1: Pure Mathematics

PMT Mock 1

	Paper Reference(s)
Time: 2 hours	**9MA0/01**

You must have:
Mathematical Formulae and Statistical Tables, calculator

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use black ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 16 questions in this paper. The total mark is 100.
- The marks for each question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.
- If you change your mind about an answer, cross it out and put your new answer and any working underneath.

1. a. Find the first four terms, in ascending powers of x, of the binomial expansion of

$$(\tfrac{1}{9} - 2x)^{\frac{1}{2}}$$

giving each coefficient in its simplest form.

(4)

b. Explain how you could use $x = \frac{1}{36}$ in the expansion to find an approximation for $\sqrt{2}$.

There is no need to carry out the calculation.

(2)

a.

$$\left(\frac{1}{9} - 2x\right)^{\frac{1}{2}} = \frac{1}{9}^{\frac{1}{2}}\left(1 - \frac{2}{1 \div 9}x\right)^{\frac{1}{2}}$$

$$= \frac{1}{3}(1 - 18x)^{\frac{1}{2}}$$

Using the binomial expansion formula:

$$= \frac{1}{3}\left(1 + \frac{(-18x)}{2} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}(-18x)^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(-18x)^3 + \cdots\right)$$

$$= \frac{1}{3}\left(1 - 9x - \frac{81}{2}x^2 - \frac{729}{2}x^3 + \cdots\right)$$

$$= \frac{1}{3} - 3x - \frac{27}{2}x^2 - \frac{243}{2}x^3 + \cdots$$

B1 For taking out a factor of $(\frac{1}{9})^{\frac{1}{2}}$

M1 For the form of the binomial expansion with $n = \frac{1}{2}$ and a term of (kx)

A1 Three of the four terms are correct

A1 cso All terms are correct. $\frac{1}{3} - 3x - \frac{27}{2}x^2 - \frac{243}{2}x^3 + \ldots$

b.

If $x = \frac{1}{36}$, $\left(\frac{1}{9} - 2x\right)^{\frac{1}{2}} = \frac{\sqrt{2}}{6}$. So $\sqrt{2}$ can be approximated by substituting $x = \frac{1}{36}$ into the expansion and multiplying by 6

M1 Score for substituting $x = \frac{1}{36}$ into $(\frac{1}{9} - 2x)^{\frac{1}{2}}$ to obtain $\frac{\sqrt{2}}{6}$ oe such as $\sqrt{\frac{2}{36}}$

A1 Explains that $x = \frac{1}{36}$ is substituted into both sides and you multiply the result by 6.

(Total for Question 1 is 6 marks)

2. The curves C_1 and C_2 have equations

$$C_1: \quad y = 2^{3x+2}$$

$$C_2: \quad y = 4^{-x}$$

Show that the x-coordinate of the point where C_1 and C_2 intersect is $\frac{-2}{5}$.

$$4^{-x} = 2^{3x+2} \Rightarrow 2^{-2x} = 2^{3x+2}$$

$$-2x = 3x + 2 \Rightarrow x = \frac{-2}{5}$$

(3)

M1 Writes 4^{-x} as a power of 2 or equivalent eg. $4^{-x} = 2^{-2x}$

Alternatively writes 2^{-2x} as a power of 4 eg. $2^{3x+2} = \left(4^{\frac{1}{2}}\right)^{(3x+2)}$

dM1 Equates the indices and attempts to find $x = \ldots$

A1 Cso $x = \frac{-2}{5}$

(Total for Question 2 is 3 marks)

3. Relative to a fixed origin,

- point A has position vector $-2\mathbf{i} + 4\mathbf{j} + 7\mathbf{k}$
- point B has position vector $-\mathbf{i} + 3\mathbf{j} + 8\mathbf{k}$
- point C has position vector $\mathbf{i} + \mathbf{j} + 4\mathbf{k}$
- point D has position vector $-\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$

a. Show that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel and the ratio $\overrightarrow{AB} : \overrightarrow{CD}$ in its simplest form.

$\overrightarrow{AB} = -\overrightarrow{OA} + \overrightarrow{OB} = 2\boldsymbol{i} - 4\boldsymbol{j} - 7\boldsymbol{k} - \boldsymbol{i} + 3\boldsymbol{j} + 8\boldsymbol{k}$

$= \boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$

$\overrightarrow{CD} = -\overrightarrow{OC} + \overrightarrow{OD} = -\boldsymbol{i} - \boldsymbol{j} - 4\boldsymbol{k} - \boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}$

$= -2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$

$\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel as $\overrightarrow{CD} = -2\overrightarrow{AB}$, and $\overrightarrow{AB}:\overrightarrow{CD} = 1:2$

M1 Attempts to subtract either way round of either $\overrightarrow{AB}$ or $\overrightarrow{CD}$

A1 Correctly obtains either $\overrightarrow{AB}$ **or** $\overrightarrow{CD}$

A1 Correctly obtains both $\overrightarrow{AB}$ **and** $\overrightarrow{CD}$

B1 States the ratio of $\overrightarrow{AB} : \overrightarrow{CD} = 1 : 2$

(4)

b. Hence describe the quadrilateral $ABCD$.

A quadrilateral with one set of parallel sides is a trapezium

(1)

B1 describes that the quadrilateral $ABCD$ is a trapezium

(Total for Question 3 is 5 marks)

4. Ben starts a new company.

- In year 1 his profits will be £24000.
- In year 11 his profit is predicted to be £64000.

Model ***P*** assumes that his profit will increase by the same amount each year.

a. According to model ***P***, determine Ben's profit in year 5.

Ben's profits can be modelled by an arithmetic progression, therefore has n^{th} term $a + (n-1)d$. $a = 24000, d = \frac{64000-24000}{10} = 4000$

$$5^{th} term: 24000 + (5-1)4000 = 40000$$

M1 Using the n^{th} term $= a + (n-1)d$ of an A.P. and attempts to find value of d

M1 Uses $a + 4d$ with $a = 24000$ and $d = \cdots (4000)$ to find the profit in Year 5

A1 £40000

(3)

Model ***Q*** assumes that his profit will increase by the same percentage each year.

b. According to model ***Q***, determine Ben's profit in year 5. Give your answer to the nearest £10.

In this case, Ben's profits can be modelled with a geometric progression, with n^{th} term ar^{n-1}.

$a = 24000 \Rightarrow 24000r^{10} = 64000 \Rightarrow r = \sqrt[10]{\frac{64000}{24000}} = \left(\frac{64}{24}\right)^{\frac{1}{10}}$

In year 5, profit$= 24000\left(\left(\frac{64}{24}\right)^{\frac{1}{10}}\right)^4 = £35530.29$

$= £35530$ to the nearest £10.

(3)

M1 Using the n^{th} term $= ar^{n-1}$ of a G.P. and attempts to find r

M1 Uses ar^4 with $a = 24000$ and $r = \left(\frac{64}{24}\right)^{\frac{1}{10}}$ to find the profit in year 5

A1 £35530

(Total for Question 4 is 6 marks)

5. The function f is defined by

$$\mathrm{f}: x \to \frac{2x-3}{x-1} \qquad x \in R,\ x \neq 1$$

a. Find $\mathrm{f}^{-1}(3)$.

$$y = \frac{2x-3}{x-1}$$

$x = \frac{2y-3}{y-1} \Rightarrow x(y-1) = 2y-3 \Rightarrow xy - x = 2y - 3 \Rightarrow xy - 2y = x - 3$

$y(x-2) = x - 3 \Rightarrow y = \frac{x-3}{x-2}$

$f^{-1}(x) = \frac{x-3}{x-2} \Rightarrow f^{-1}(3) = \frac{3-3}{3-2} = 0$

M1 For either attempting to solve $\frac{2x-3}{x-1} = 3$ leading to a value of x or score for substituting in $x = 3$ in $\mathrm{f}^{-1}(x)$ where $\mathrm{f}^{-1}(x) = \frac{x-3}{x-2}$

A1 $f^{-1}(3) = 0$

(2)

b. Show that

$$\mathrm{ff}(x) = \frac{x+p}{x-2} \qquad x \in R,\ x \neq 2$$

where p is an integer to be found.

$ff(x) = \frac{2\left(\frac{2x-3}{x-1}\right)-3}{\left(\frac{2x-3}{x-1}\right)-1} \Rightarrow ff(x) = \frac{\frac{4x-6}{x-1}-\frac{3x-3}{x-1}}{\frac{2x-3}{x-1}-\frac{x-1}{x-1}}$

$= \frac{4x-6-3x+3}{2x-3-x+1} = \frac{x-3}{x-2}$

M1 For an attempt substituting $\frac{2x-3}{x-1}$ in f(x).

dM1 Attempts to multiply all terms on the numerator and denominator by $(x-1)$ to obtain a fraction $\frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are linear expressions.

A1 cso $\frac{x-3}{x-2}$ with $p = -3$

(3)

The function g is defined by

$$g: x \to x^2 - 5x \qquad x \in R,\ \ 0 \leq x \leq 6$$

c.Find the range of g.

$\frac{dg}{dx} = 2x - 5$
$2x - 5 = 0 \Rightarrow x = \frac{5}{2}$
$g(6) = 36 - 30 = 6$
$-\frac{25}{4} \leq g(x) \leq 6$

c. M1 Either applies the completing the square method to establish the minimum of g. Or differentiating the quadratic, setting the result equal to zero, finding x and inserting this value of x back into g(x) in order to find the minimum.

B1 For either finding the correct minimum or maximum value of g

A1 $-\frac{25}{4} \leq g(x) \leq 6$ or $-\frac{25}{4} \leq g \leq 6$ or $-\frac{25}{4} \leq y \leq 6$

(3)

d. Explain why the function g does not have an inverse.

$g(x)$ does not have an inverse as it is not one-to-one

B1 either the function g is many-one or the function g is not one-one

(1)

(Total for Question 5 is 9 marks)

6. a. Express $4\sin x - 5\cos x$ in the form $R\sin(x-\alpha)$, where $R > 0$ and $0 < \alpha < 90^0$.

Give the exact value of R, and give the value of α, in degrees, to 2 decimal places.

$R = \sqrt{4^2+5^2} = \sqrt{41}$ only

$\tan\alpha = \frac{5}{4} \Rightarrow \alpha = 51.34°$

B1 $R = \sqrt{4^2+5^2} = \sqrt{41}$ only

M1 Proceeds to a value of α from $\tan\alpha = \pm\frac{5}{4}, \tan\alpha = \pm\frac{4}{5},\ or\ \cos\alpha = \pm\frac{4}{R}$

A1 $\alpha = 51.34°$ or 0.8961 radians

(3)

$$T = \frac{8400}{19 + (4\sin x - 5\cos x)^2}\ ,\ x > 0$$

b. Use your answer to part *a* to calculate

i. the minimum value of T,

$T = \frac{8400}{19+(\sqrt{41})^2} = \frac{8400}{60} = 140$

M1 for an attempt at $\frac{8400}{19+(R)^2}$

A1 140

ii. the smallest value of x, $x > 0$, at which this minimum value occurs.

M1 Uses $x -$ their $\alpha = (2n+1)90^0$ to find x.

e.g. $90^0 \pm 51.34^0$

A1 141.34^0

(4)

(Total for Question 6 is 7 marks)

7.

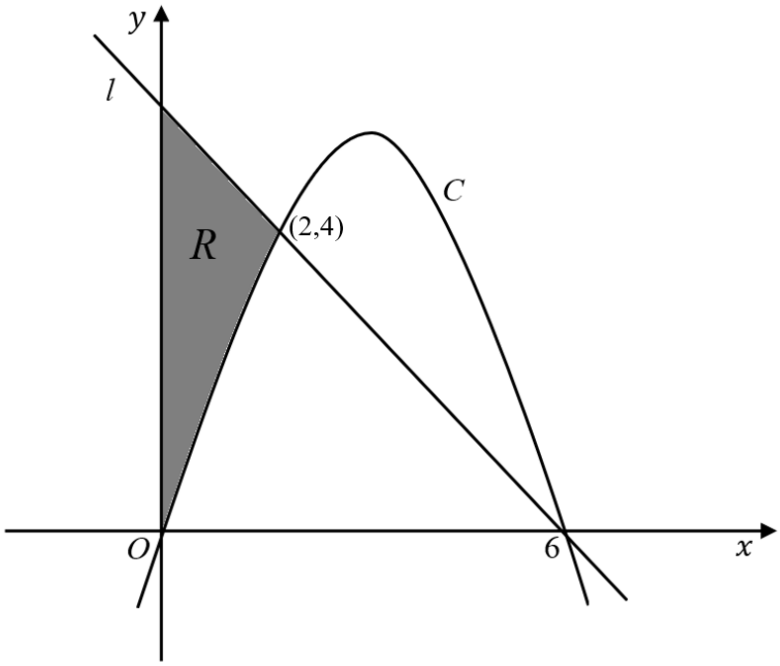


Figure 1

Figure 1 shows a sketch of a curve C with equation $y = \mathrm{f}(x)$ and a straight line l.

The curve C meets l at the points (2,4) and (6,0) as shown.

The shaded region R, shown shaded in Figure 1, is bounded by C, l and the y-axis.

Given that $\mathrm{f}(x)$ is a quadratic function in x, use inequalities to define region R.

Working out equation of line l:

Gradient is given by: $\frac{4}{-4} = -1$

$$y - 4 = -1(x - 2) \Rightarrow y = -x + 6$$

Given that C is a quadratic function it has equation $y = ax^2 + bx$ (no c as the y intercept is 0)

Substituting in (2,4) and (6,0) gives:

$$4 = 4a + 2b, 0 = 36a + 6b$$

Solving simultaneously, this gives:

$$a = -\frac{1}{2}, b = 3$$

Therefore $C: y = -\frac{1}{2}x^2 + 3x$

The region R is defined by $x \geq 0,\ \ y \leq -x + 6,\ \ y \geq \frac{1}{2}x(6 - x)$

M1 Attempts to find the gradient of equation of line l with points (2,4) and (6,0) and substitutes either (2,4) or (6,0) into $y - y_1 = m(x - x_1)$ to obtain an equation of line l

A1 $y = -x + 6$

M1 A complete method to find the constant a in $y = ax(6 - x)$ or the constants a, b in $y = ax^2 + bx$, $a = -\frac{1}{2}$, $b = 3$

A1 Equation of the curve C is $y = \frac{1}{2}x(6 - x)$ or $y = -\frac{1}{2}x^2 + 3x$

B1 Fully defines the region R.

$x \geq 0$, $y \leq -x + 6$, $y \geq \frac{1}{2}x(6 - x)$

(5)

(Total for Question 7 is 5 marks)

8.

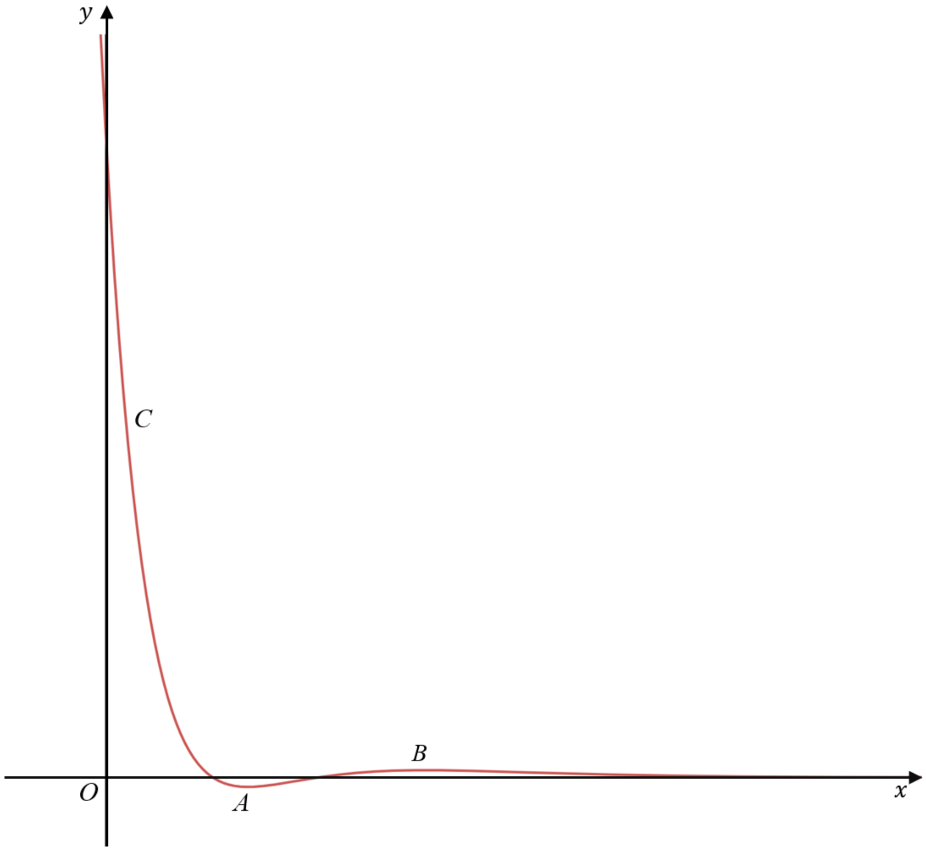


Figure 2

Figure 2 shows a sketch of the curve C with the equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = (2x^2 - 9x + 9)e^{-x}, \quad x \in R$$

The curve has a minimum turning point at A and a maximum turning point at B as shown in the figure above.

a. Find the coordinates of the point where C crosses the y–axis.

$$f(0) = (2(0) - 9(0) + 9)e^{-0}$$

$$= 9$$

Therefore the coordinates are $(0,9)$

(1)

b. Show that $f'(x) = -(2x^2 - 13x + 18)e^{-x}$

The product rule states that:

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

And gives:

$$f'(x) = (4x - 9)e^{-x} - (2x^2 - 9x + 9)e^{-x}$$
$$= -(2x^2 - 13x + 18)e^{-x}$$

M1 Attempts the product rule or quotient and uses $e^{-x} \to ke^{-x}$, $k \neq 0$

A1 A correct $f'(x)$ which may be unsimplified.

$f'(x) = (4x - 9)e^{-x} - (2x^2 - 9x + 9)e^{-x}$ or $f'(x) = \frac{e^x(4x-9)+(2x^2-9x+9)e^x}{e^{2x}}$

A1 Proceeds correctly to given answer showing all necessary steps.

$f'(x) = -(2x^2 - 13x + 18)e^{-x}$

(3)

c. Hence find the exact coordinates of the turning points of C.

Turning points of C are given by

$$f'(x) = 0 \Rightarrow -(2x^2 - 13x + 18)e^{-x} \Rightarrow 2x^2 - 13x + 18 = 0$$

$$x = \frac{9}{2}, x = 2$$

Therefore $y = -e^{-2}$ or $y = 9e^{-\frac{9}{2}}$, and the stationary points are given by $(2, -e^{-2})$ and $\left(\frac{9}{2}, 9e^{-\frac{9}{2}}\right)$.

B1 States the roots of $f'(x) = 0$ as $2x^2 - 13x + 18 = 0 \Rightarrow x = 2, \frac{9}{2}$

M1 Substitutes either $x = 2$ or $x = \frac{9}{2}$ into $f(x)$ to find a y value.

A1 Obtains $(2, -e^{-2})$ and $(\frac{9}{2}, 9e^{-\frac{9}{2}})$ as the stationary points.

(3)

The graph with equation $y = \mathrm{f}(x)$ is transformed onto the graph with equation

$$y = a\mathrm{f}(x) + b\,, \quad x \geq 0$$

The range of the graph with equation $y = a\mathrm{f}(x) + b$ is $0 \leq y \leq 9e^2 + 1$

Given that a and b are constants.

d. find the value of a and the value of b.

The curve C is stretched vertically with scale factor a, and vertically translated up b units.

The y-intercept of C is (9,0), which will be the maximum value for $x \geq 0$. Therefore 9 will become $9e^2 + 1 \Rightarrow a = e^2, b = 1$

(2)

B1 Either $a = e^2$ or $b = 1$

B1 Both $a = e^2$ and $b = 1$

(Total for Question 8 is 9 marks)

9. a. Use the substitution $t^2 = 2x - 5$ to show that

$$\int \frac{1}{x+3\sqrt{2x-5}} \, \mathrm{d}x = \int \frac{2t}{t^2+6t+5} \, \mathrm{d}t$$

$$t^2 = 2x - 5 \Rightarrow 2t\frac{dt}{dx} = 2 \Rightarrow tdt = dx$$

$$\int \frac{1}{\frac{t^2+5}{2}+3t} dx = \int \frac{1}{\frac{t^2+5}{2}+3t} tdt$$

$$= \int \frac{2t}{t^2+5+6t} \, dt$$

B1 $t\mathrm{d}t = \mathrm{d}x$ or equivalent

M1 Attempts a full substitution of $t^2 = 2x - 5$ and $x = \frac{t^2+5}{2}$, including $\mathrm{d}x = t\mathrm{d}t$ to form an integrand in terms of t

A1 Clear reasoning including one fully correct intermediate line, including the integral signs, leading to the given expression.

(3)

b. Hence find the exact value of

$$\int_3^{27} \frac{1}{x+3\sqrt{2x-5}} \, \mathrm{d}x$$

Using partial fractions:

$$\frac{2t}{t^2+6t+5} = \frac{2t}{(t+5)(t+1)} = \frac{A}{t+5} + \frac{B}{t+1}$$

$$2t = A(t+1) + B(t+5)$$

$$t = -5 \Rightarrow -10 = -4A \Rightarrow A = \frac{5}{2}$$

$$t = -1 \Rightarrow -2 = 4B \Rightarrow B = -\frac{1}{2}$$

$$\int \frac{1}{x+3\sqrt{2x-5}} = \int \frac{5}{2(t+5)} - \frac{1}{2(t+1)}$$

$$= \frac{5\ln(x+5)}{2} - \frac{\ln(x+1)}{2} + c$$

Finding the new limits:

$$x = 27 \Rightarrow t = 7, x = 3 \Rightarrow t = 1$$

$$\int_3^{27} \frac{1}{x + 3\sqrt{2x-5}} \, \mathrm{d}x = \left[\frac{5\ln(x+5)}{2} - \frac{\ln(x+1)}{2}\right]_1^7$$

$$= \left[\frac{5}{2}\ln(7+5) - \frac{1}{2}\ln(7+1)\right] - \left[\frac{5}{2}\ln(1+5) - \frac{1}{2}\ln(1+1)\right] = \frac{5}{2}\ln(2) + \frac{1}{2}\ln\left(\frac{1}{4}\right)$$

$$\ln\sqrt{8}$$

(5)

b. M1 Uses correct form of Partial Fraction leading to values of A and B

A1 Correct Partial Fraction $\frac{2t}{t^2+6t+5} = \frac{\frac{5}{2}}{t+5} + \frac{-\frac{1}{2}}{t+1}$

dM1 Integrates using lns. e.g. $P\ln(t+5) + Q\ln(t+1) \Rightarrow \frac{5}{2}\ln(t+5) - \frac{1}{2}\ln(t+1)$

M1 Uses either the limits 7 and 1 with their attempted integral or alternatively substitutes $t = (2x-5)^{\frac{1}{2}}$ and uses the limits 27 and 3 within their attempted integral. Applies the addition law or subtraction law leading to the form $k\ln a$ or $\ln b$ where a and b are constants.

A1 $\ln 2^{\frac{3}{2}}$ or $\ln\sqrt{8}$

(Total for Question 9 is 8 marks)

10.

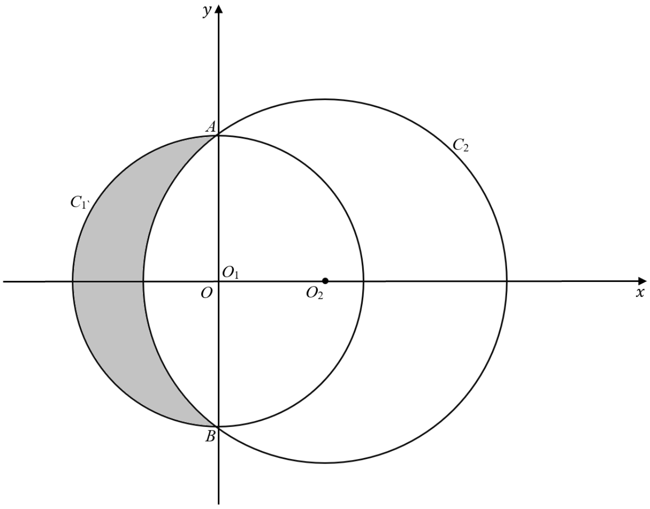


Figure 3

Circle C_1 has equation $x^2 + y^2 = 64$ with centre O_1.

Circle C_2 has equation $(x - 6)^2 + y^2 = 100$ with centre O_2.

The circles meet at points A and B as shown in Figure 3.

a. Show that angle $AO_2B = 1.85$ radians to 3 significant figures.

$O_2 = (6.0)$
$AO_2 = BO_2 = 10$
$AB = 16$, as it is the diameter of C_1
$\frac{AO_2B}{2} = \sin^{-1}\frac{8}{10} = 0.927$
$AO_2B = 1.8545\ rad$
$= 1.85$ to 3 sig fig as required

B1 C_1 has centre (0,0), radius = 8 and C_2 has centre (6,0), radius 10

M1 Uses the radii of the circles C_1 and C_2 and correct attempt to find angle AO_2B in circle C_2.

e.g. Attempts $\sin AO_2O = \frac{8}{10}$ to find AO_2O then $\times 2$

Alternatively uses $\cos AO_2O = \frac{6}{10}$ to find AO_2O then $\times 2$

Or uses $\tan AO_2O = \frac{8}{6}$ to find AO_2O then $\times 2$

OR uses cosine rule $\cos AO_2B = \frac{10^2+10^2-16^2}{2\times10\times10} = -\frac{56}{200} \Rightarrow AO_2B = \cos^{-1}\left(-\frac{56}{200}\right) = \cdots$

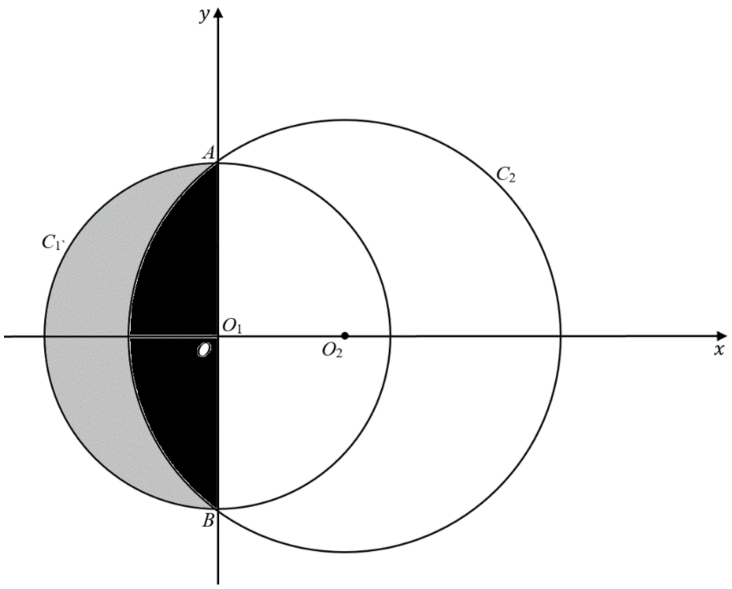
A1 Correct and careful work in proceeding to the given answer.

i.e. 1.85 radians

(3)

b. Find the area of the shaded region, giving your answer correct to 1 decimal place.

(3)



Area of sector AO_2B $-$ area of triangle AO_2B $=$

$$= \frac{1}{2} \times 10^2 \times (1.85) - \frac{1}{2} \times 10^2 \times \sin 1.85$$

$=44.436$

Area of the region shaded grey $=$ Area of semicircle with centre O_1 – Area of segment

$$= \frac{\pi \times 8^2}{2} - 44.436$$

$$= 56.1$$

b. M1 Attempts to use the correct formula to find the area of the segment shaded black with centre O_2.

M1 Attempts to use the correct method in order to find area of the region shaded grey.

A1 56.1

(Total for Question 10 is 6 marks)

11. In a science experiment, a radio active particle, N, decays over time, t, measured in minutes. The rate of decay of a particle is proportional to the number of particles remaining.

Write down a suitable equation for the rate of change of the number of particles, N in terms of t.

M1 Any equation of the correct form, involving N and an exponential in t.

e.g. $N = e^{\pm t}$, $N = Ae^{\pm t}$, $N = Ae^{\pm kt}$

Allow $\ln N = kt + c$

A1 $N = Ae^{-kt}$

(2)

(Total for Question 11 is 2 marks)

12. a. Show that

$$\sec\theta - \cos\theta = \sin\theta\tan\theta \qquad \theta \neq (\pi n)^0 \quad n \in Z$$

$$\frac{1}{\cos\theta} - \cos\theta = \frac{1-\cos^2\theta}{\cos\theta} = \frac{\sin^2\theta}{\cos\theta} = \sin\theta\tan\theta$$

B1 States or uses $\sec\theta = \frac{1}{\cos\theta}$

M1 Attempts to obtain a single fraction.

A1 Shows all the necessary steps leading to given answer.

(3)

b. Hence, or otherwise, solve for $0 < x \leq \pi$

$$\sec x - \cos x = \sin x \tan(3x - \tfrac{\pi}{9})$$

$$\sin x \tan x = \sin x \tan\left(3x - \frac{\pi}{9}\right)$$

$$\tan x = \tan\left(3x - \frac{\pi}{9}\right)$$

$$x = 3x - \frac{\pi}{9} \Rightarrow x = \frac{\pi}{18}$$

Second solution can be found from $x + \pi = 3x - \frac{\pi}{9} \Rightarrow x = \frac{5\pi}{9}$

Third solution can be found from $\sin x = 0 \Rightarrow x = \pi$

(5)

b. M1 Uses part (a), cancels or factorises out the $\sin x$ term, to establish that one solution is found when $x = 3x - \frac{\pi}{9}$.

A1 $x = \frac{\pi}{18}$

M1 Second solution can be found by solving $x + \pi = 3x - \frac{\pi}{9}$.

A1 $x = \frac{5\pi}{9}$

B1 Deduces that a solution can be found from $\sin x = 0 \Rightarrow x = \pi$

(Total for Question 12 is 8 marks)

13. A sequence $a_1, a_2 a_3, \ldots$ is defined by

$$a_{n+1} = 5 - pa_n \qquad n \geq 1$$

where $p \in \mathbb{Z}$.

Given that

- $a_1 = 4$
- the sequence is a periodic sequence of order 2.

a. Write down an expression for a_2 and a_3.

$a_2 = 5 - p(a_1) = 5 - pa_1$
$a_3 = 5 - p(a_2) = 5 - p(5 - pa_1) = 5 - 5p + 4p^2$

M1 Applies the sequence formula $a_{n+1} = 5 - pa_n$ to find a_2 and a_3.

A1 Both are correct $a_2 = 5 - 4p$ and $a_3 = 5 - 5p + 4p^2$

(2)

b. Find the value of p.

As the sequence is of order 2, $a_1 = a_3$, therefore $4 = 5 - 5p + 4p^2 \Rightarrow p = 1, p = \frac{1}{4}$, so we choose $p = 1$.

M1 Sets $a_3 = 4$ and attempts to find the value of p

A1 $p = 1$

(2)

c. Find $\sum_{r=1}^{21} a_r$

$$\sum_{r=1}^{21} a_r = a_1 + a_2 + a_3 + a_4 + \cdots + a_{19} + a_{20} + a_{21}$$
$$= 4 + 1 + 4 + 1 + \cdots + 4 + 1 + 4 = 10 \times (4 + 1) + 4$$

(2)

M1 Uses a clear strategy to find the sum to 21 terms.

A1 54

(Total for Question 13 is 6 marks)

14. A circular stain is growing.

The rate of increase of its radius is inversely proportional to the square of the radius.

At time t seconds the circular stain has radius r cm and area A cm^2.

a. Show that $\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{k}{\sqrt{A}}$.

$\frac{dr}{dt} \propto \frac{1}{r^2} \Rightarrow \frac{dr}{dt} = \frac{c}{r^2}$

Using the chain rule, $\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}t}$. $A = \pi r^2 \Rightarrow \frac{dA}{dr} = 2\pi r$

$\frac{dA}{dt} = 2\pi r \times \frac{c}{r^2}$

Using $r = \sqrt{\frac{A}{\pi}}$ gives:

$\frac{dA}{dt} = 2\pi\sqrt{\frac{A}{\pi}} \times \frac{c}{\frac{A}{\pi}}$

$= \frac{2\pi\sqrt{\pi}c}{\sqrt{A}} = \frac{k}{\sqrt{A}}$

B1 Uses the model to state $\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{c}{r^2}$

M1 Attempts to use $\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}t}$ with $A = \pi r^2$ and $\frac{\mathrm{d}A}{\mathrm{d}r} = 2\pi r$

M1 Substitutes $r = \sqrt{\frac{A}{\pi}} = \frac{\sqrt{A}}{\sqrt{\pi}}$ into $\frac{\mathrm{d}A}{\mathrm{d}t}$ and proceeds to an expression in terms of r for

A1 Proceeds to the given answer with accurate work showing all necessary steps.

(4)

Given that

- the initial area of the circular stain is 0.09 cm^2.
- after 10 seconds the area of the circular stain is 0.36 cm^2.

b. Solve the differential equation to find a complete equation linking A and t.

We have $\frac{dA}{dt} = \frac{k}{\sqrt{A}}$. Separation of variables gives $\int \sqrt{A}\ \mathrm{d}A = \int k\ \mathrm{d}t$.

Completing the integration gives:

$\frac{2}{3}A^{\frac{3}{2}} = kt + c.$

From the question, at time $t = 0, A = 0.09$

$\frac{2}{3}(0.09)^{\frac{3}{2}} = k(0) + c \Rightarrow \frac{9}{500}$

At time $t = 10, A = 0.36$

$\frac{2}{3}(0.36)^{\frac{3}{2}} = k(10) + \frac{9}{500} \Rightarrow 10k = \frac{18}{125} - \frac{9}{500} \Rightarrow\ k = \frac{63}{5000}$

Therefore, an equation linking A and t is

$A = (\frac{189t}{10000} + \frac{27}{1000})^{\frac{2}{3}}.$

(6)

B1 Separates the variables $\int \sqrt{A}\ \mathrm{d}A = \int k\ \mathrm{d}t$

M1 Integrating the lhs and rhs.

A1 Correct integration

B1 Substitutes $t = 0, A = 0.09 \Rightarrow c = \frac{9}{500}$ or equivalent

M1 Substitutes $t = 10, A = 0.36$ to find k.

A1 Obtains any correct equation for the model.

e.g. $A = (\frac{189t}{10000} + \frac{27}{1000})^{\frac{2}{3}}$ or equaivalent

(Total for Question 14 is 10 marks)

15. The curve C has equation

$$y = \frac{1}{2}x - \frac{1}{4}\sin 2x \qquad 0 < x < \pi$$

a. Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \sin^2 x$

$\frac{d}{dx}\left(\frac{1}{2}x - \frac{1}{4}\sin 2x\right) = \frac{1}{2} - \frac{1}{2}\cos 2x$
Using the identity $\cos 2x = 1 - 2\sin^2 x$:
$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2} - \frac{1}{2}(1 - 2\sin^2 x) = \sin^2 x$.

M1 Attempts to differentiate $y = \frac{1}{2}x - \frac{1}{4}\sin 2x$ to give $\frac{\mathrm{d}y}{\mathrm{d}x} = p - q\cos 2x$ where p and q are constants.

A1 Correct differentiation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2} - \frac{1}{2}\cos 2x$

A1 Proceeds correctly to the given answer replacing $\cos 2x = 1 - 2\sin^2 x$

(3)

b. Find the coordinates of the points of inflection of the curve.

Points of inflection are found when the second derivative equals zero:

$\frac{d^2y}{dx^2} = \frac{d}{dx}(\sin^2 x) = 2\cos x \sin x$

$2\sin x \cos x = 0 \Rightarrow \sin x = 0 \text{ or } \cos x = 0 \Rightarrow x = 2k\pi \ \ or \ \ x = \frac{(2k+1)\pi}{2}, k \in \mathbb{Z}.$

For $0 < x < \pi$, this means $x = \frac{\pi}{2}$.

Points of inflection at $\left(\frac{\pi}{2}, \frac{\pi}{4}\right)$

(3)

M1 Attempts to differentiate $\frac{\mathrm{d}y}{\mathrm{d}x}$ and equate to zero and proceed to find x.

A1 $x = \frac{\pi}{2}$

A1 Fully correct substitution of $x = \frac{\pi}{2}$ in $y = \frac{1}{2}x - \frac{1}{4}\sin 2x$, point of inflection $\left(\frac{\pi}{2}, \frac{\pi}{4}\right)$

(Total for Question 15 is 6 marks)

16. Use algebra to prove that the product of any two consecutive odd numbers is an odd number.

We can write an odd number at $2k + 1$, and the next consecutive one as $2k + 3$:

$(2k + 1)(2k + 3) = 4k^2 + 8k + 3$

$4k^2 + 8k + 3 = 2(2k^2 + 4k + 1) + 1$ therefore is odd

(4)

B1 Writes any two odd numbers in the form either $2k - 1, 2k + 1$ or $2k + 1, 2k + 3$

M1 Multiplying out the two brackets.

A1 Correct expression for multiplying out the two consecutive odd numbers.

A1 Correct conclusion drawn from fully correct working.

(4)

(Total for Question 16 is 4 marks)