Please check the examination details below before entering your candidate information

Candidate surname	Other names

Pearson Edexcel Level 3 GCE

Centre Number	Candidate Number

Wednesday 7 October 2020

Morning (Time: 2 hours)	Paper Reference **9MA0/01**

Mathematics

Advanced
Paper 1: Pure Mathematics 1

You must have: Mathematical Formulae and Statistical Tables (Green), calculator	Total Marks

Candidates may use any calculator allowed by Pearson regulations. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions
- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need*.
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information
- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 16 questions in this question paper. The total mark for this paper is 100.
- The marks for each question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice
- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over

1. (a) Find the first four terms, in ascending powers of x, of the binomial expansion of

$$(1+8x)^{\frac{1}{2}}$$

giving each term in simplest form.

(3)

a) Find the first four terms of binomial expansion of $(1+8x)^{1/2}$

General Formula : $(1+y)^n = 1 + \frac{ny}{1!} + \frac{n(n-1)y^2}{2!} + \frac{n(n-1)(n-2)y^3}{3!} + \dots$ (four terms)

$(1+8x)^{1/2} \Rightarrow y = 8x$ and $n = \frac{1}{2}$ 1

$(1+8x)^{1/2} = 1 + \frac{\frac{1}{2}\times 8x}{1!} + \frac{\frac{1}{2}(\frac{1}{2}-1)(8x)^2}{2!} + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)(8x)^3}{3!} + \dots$ 1

$(1+8x)^{1/2} = \underline{1 + 4x - 8x^2 + 32x^3 + \dots}$ 1

(b) Explain how you could use $x = \frac{1}{32}$ in the expansion to find an approximation for $\sqrt{5}$

There is no need to carry out the calculation.

(2)

b) • We should substitute $x = \frac{1}{32}$ into $(1+8x)^{1/2}$ and this will give $\frac{\sqrt{5}}{2}$ 1

• We should then substitute $x = \frac{1}{32}$ into $1+4x-8x^2+32x^3$ and we then multiply the result by 2 to give $\underline{\sqrt{5}}$. 1

2. By taking logarithms of both sides, solve the equation

$$4^{3p-1} = 5^{210}$$

giving the value of p to one decimal place.

(3)

$4^{3p-1} = 5^{210} \Rightarrow \log(4^{3p-1}) = \log(5^{210})$

$\Rightarrow (3p-1)\log(4) = 210\log(5)$ 1

$\Rightarrow 3p-1 = \dfrac{210\log(5)}{\log(4)}$

$\Rightarrow 3p = 243.80245 + 1$

$\Rightarrow p = 81.6008...$ 1

$\Rightarrow p = \underline{81.6}$ (1 d.p) 1

log laws

$\log(a^b) \Rightarrow b\log(a)$

3. Relative to a fixed origin O

- point A has position vector $2\mathbf{i} + 5\mathbf{j} - 6\mathbf{k}$
- point B has position vector $3\mathbf{i} - 3\mathbf{j} - 4\mathbf{k}$
- point C has position vector $2\mathbf{i} - 16\mathbf{j} + 4\mathbf{k}$

(a) Find $\overrightarrow{AB}$

(2)

a) $\overrightarrow{AB} = B - A$

$A = \begin{pmatrix} 2 \\ 5 \\ -6 \end{pmatrix}, B = \begin{pmatrix} 3 \\ -3 \\ -4 \end{pmatrix}$

$\Rightarrow \overrightarrow{AB} = \begin{pmatrix} 3 \\ -3 \\ -4 \end{pmatrix} - \begin{pmatrix} 2 \\ 5 \\ -6 \end{pmatrix} = \begin{pmatrix} 1 \\ -8 \\ 2 \end{pmatrix}$ 1

$\Rightarrow \overrightarrow{AB} = \underline{\underline{\mathbf{i} - 8\mathbf{j} + 2\mathbf{k}}}$ 1

(b) Show that quadrilateral $OABC$ is a trapezium, giving reasons for your answer.

(2)

b) $\overrightarrow{AB} = \begin{pmatrix} 1 \\ -8 \\ 2 \end{pmatrix}$ and $\overrightarrow{OC} = \begin{pmatrix} 2 \\ -16 \\ 4 \end{pmatrix}$

$\Rightarrow \overrightarrow{OC} = 2\overrightarrow{AB}$ 1

$\Rightarrow \overrightarrow{OC}$ is parallel to $\overrightarrow{AB}$ $\Rightarrow$ OABC is a trapezium. 1

4. The function f is defined by

$$f(x) = \frac{3x-7}{x-2} \qquad x \in \mathbb{R}, x \neq 2$$

(a) Find $f^{-1}(7)$

(2)

a) $f(x) = \frac{3x-7}{x-2} \Rightarrow y = \frac{3x-7}{x-2}$

1. Swap x and y
2. Solve for y

$\Rightarrow x = \frac{3y-7}{y-2} \Rightarrow x(y-2) = 3y-7$

$\Rightarrow xy - 2x = 3y - 7$

$\Rightarrow xy - 3y = 2x - 7$

$\Rightarrow y(x-3) = 2x-7$

$\Rightarrow y = \frac{2x-7}{x-3}$

$\Rightarrow f^{-1}(x) = \frac{2x-7}{x-3} \Rightarrow f^{-1}(7) = \frac{2(7)-7}{7-3} = \frac{7}{4}$ 1

$\Rightarrow f^{-1}(7) = \frac{7}{4}$ 1

(b) Show that $ff(x) = \frac{ax+b}{x-3}$ where a and b are integers to be found.

(3)

b)

$f(x) = \frac{3x-7}{x-2}$

$\Rightarrow ff(x) = \frac{3f(x)-7}{f(x)-2}$

$= \frac{2x-7}{x-2} \times \frac{x-2}{x-3}$ 1

$\Rightarrow ff(x) = \frac{2x-7}{x-3} = \frac{ax+b}{x-3}$ as required with $a = 2$ and $b = -7$. 1

1

$3f(x) = 3\left(\frac{3x-7}{x-2}\right) = \frac{9x-21}{x-2}$

$3f(x) - 7 = \frac{9x-21}{x-2} - \frac{7}{1} = \frac{9x-21-7x+14}{x-2}$

$= \frac{2x-7}{x-2}$ (numerator)

$f(x) - 2 = \frac{3x-7}{x-2} - \frac{2}{1} = \frac{3x-7-2x+4}{x-2}$

$= \frac{x-3}{x-2}$ (denominator)

5. A car has six forward gears.

The fastest speed of the car

- in 1st gear is 28 km h^{-1}
- in 6th gear is 115 km h^{-1}

Given that the fastest speed of the car in successive gears is modelled by an **arithmetic sequence**,

(a) find the fastest speed of the car in 3rd gear.

(3)

a) Arithmetic Sequence : $a_n = a + (n-1)d$

a_n = n'th term
a = first / initial term (28 kmh^{-1})
d = Common difference between terms.

$a = 28$, $a_6 = 115$

$\Rightarrow \quad a_6 = 115 = 28 + (6-1) \cdot d$

$\Rightarrow \quad 5d = 115 - 28 \quad \Rightarrow \quad d = \frac{115-28}{5} = \underline{17.4}$ 1

$\Rightarrow \quad a_3 = 28 + (3-1)\,17.4$ 1

$\Rightarrow \quad a_3 = \underline{62.8 \; kmh^{-1}}$ is the fastest speed of the car in 3rd gear. 1

Given that the fastest speed of the car in successive gears is modelled by a **geometric sequence**,

(b) find the fastest speed of the car in 5th gear.

(3)

b) Geometric Sequence : $a_n = ar^{n-1}$

a_n = n'th term
a = first / initial term
r = Common ratio between terms

$a_6 = 115 \; kmh^{-1}$ and $a = 28 \; kmh^{-1}$

$\Rightarrow \quad a_6 = 115 = 28 \cdot r^5$

$\Rightarrow \quad r^5 = \frac{115}{28} \quad \Rightarrow \quad r = \left(\frac{115}{28}\right)^{1/5} = 1.3265\ldots$ 1

$\Rightarrow \quad a_5 = 28 \cdot (1.3265\ldots)^4 = 86.6941\ldots$ 1 $\Rightarrow \quad a_5 = \underline{86.7} \; kmh^{-1}$ is the fastest Speed of the car in 5th gear. 1

6. (a) Express $\sin x + 2\cos x$ in the form $R\sin(x+\alpha)$ where R and α are constants, $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R and give the value of α in radians to 3 decimal places.

(3)

a) $\sin x + 2\cos x \longrightarrow R\sin(x+\alpha)$

$\boxed{1}$ Find α
$\boxed{2}$ Find R

$R\sin(x+\alpha) = R\sin x\cos\alpha + R\cos x\sin\alpha \Rightarrow \sin x = R\sin x\cos\alpha \Rightarrow R\cos\alpha = 1$

$2\cos x = R\cos x\sin\alpha \Rightarrow R\sin\alpha = 2$

$\Rightarrow \tan\alpha = \frac{2}{1} \Rightarrow \alpha = \tan^{-1}(2)$

$\alpha = 1.10714\ldots$ 1 $\Rightarrow \alpha = \underline{1.107}$ (3 d.p) 1

$R = \sqrt{(1)^2 + (2)^2} = \sqrt{5}$ 1

$\Rightarrow \alpha = \underline{1.107}$ (radians), $R = \underline{\sqrt{5}} \Rightarrow \sin x + 2\cos x = \underline{\sqrt{5}\sin(x+1.107)}$

The temperature, θ°C, inside a room on a given day is modelled by the equation

$$\theta = 5 + \sin\left(\frac{\pi t}{12} - 3\right) + 2\cos\left(\frac{\pi t}{12} - 3\right) \qquad 0 \leqslant t < 24$$

where t is the number of hours after midnight.

Using the equation of the model and your answer to part (a),

(b) deduce the maximum temperature of the room during this day,

(1)

b) $\theta = 5 + \sin\left(\frac{\pi t}{12} - 3\right) + 2\cos\left(\frac{\pi t}{12} - 3\right)$

let $x = \frac{\pi t}{12} - 3$, we can use our answer from part a. $\left(\sqrt{5}\sin(x+1.107) = \sin x + 2\cos x\right)$

$\Rightarrow \theta = 5 + \sqrt{5}\sin\left(\frac{\pi t}{12} - 3 + 1.107\right)$, we have a maximum when $\sin x = 1$

$\Rightarrow \theta = (5+\sqrt{5})^\circ C$ or $\theta = \underline{7.24}\ ^\circ C$ (3 s.f) 1

(c) find the time of day when the maximum temperature occurs, giving your answer to the nearest minute.

(3)

c) $\theta = 5 + \sqrt{5}\sin\left(\frac{\pi t}{12} - 3 + 1.107\right)$

In part b, we said the maximum temperature occurs when $\sin x = 1$.

$\Rightarrow x = \sin^{-1}(1)$

$\Rightarrow x = \underline{\pi/2}$

$\Rightarrow \frac{\pi t}{12} - 3 + 1.107 = \frac{\pi}{2}$

$\Rightarrow \frac{\pi t}{12} = \frac{\pi}{2} + 3 - 1.107$

0.2 of an hour is equal $0.2 \times 60 = 12$ mins

$\Rightarrow \cancel{\pi}t = \frac{12\left(\frac{\pi}{2} + 3 - 1.107\right)}{\pi}$ 1 $\Rightarrow t = 13.2$ hours 1

$\Rightarrow t = 13$ hours and 12 minutes after midnight. 1

7.

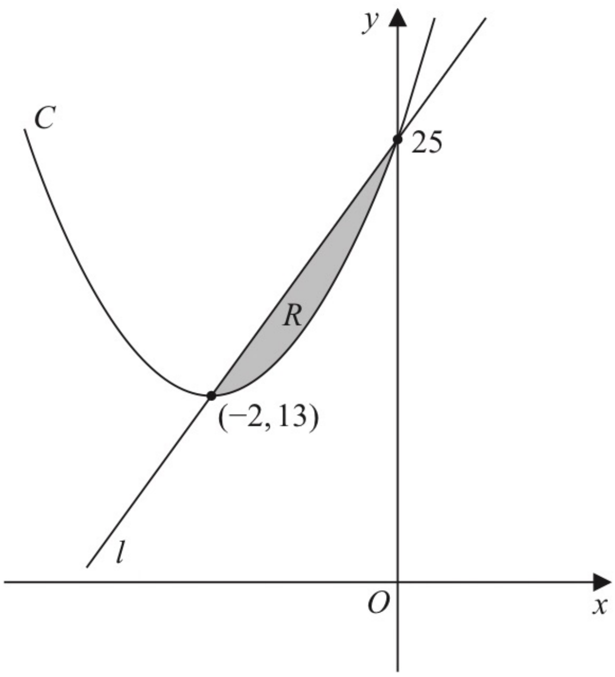


Figure 1

Figure 1 shows a sketch of a curve C with equation $y = f(x)$ and a straight line l.

The curve C meets l at the points $(-2, 13)$ and $(0, 25)$ as shown.

The shaded region R is bounded by C and l as shown in Figure 1.

Given that

- $f(x)$ is a quadratic function in x
- $(-2, 13)$ is the minimum turning point of $y = f(x)$

use inequalities to define R.

(5)

a) $l: y = mx + c$, $c = 25$ (y-intercept on graph)

We will use the point $(-2, 13)$ to work out m.

$(-2, 13)$: $13 = -2m + 25 \Rightarrow 2m = 12$ ①

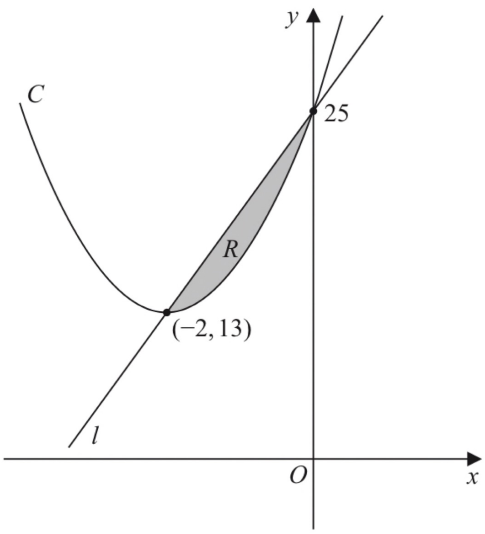
$\Rightarrow m = 6 \Rightarrow l: y = 6x + 25$ ①

$\Rightarrow f(x) = a(x+2)^2 + 13$

$\Rightarrow (0, 25)$. $25 = 4a + 13 \Rightarrow 4a = 12$, thus $a = 3$ ①

$\Rightarrow C: y = 3(x+2)^2 + 13$ ①

$\Rightarrow 3(x+2)^2 + 13 < y < 6x + 25$ ①



8. A new smartphone was released by a company.

The company monitored the total number of phones sold, n, at time t days after the phone was released.

The company observed that, during this time,

the rate of increase of n was proportional to n

Use this information to write down a suitable equation for n in terms of t.

(*You do not need to evaluate any unknown constants in your equation.*)

(2)

$n = Ae^{kt}$ 2 A and k are both positive constants.

We want an equation which is to do with exponential growth.

9.

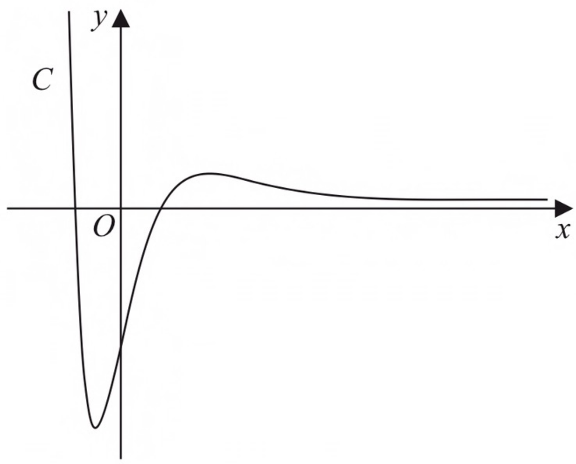


Figure 2

Figure 2 shows a sketch of the curve C with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = 4(x^2 - 2)\mathrm{e}^{-2x} \qquad x \in \mathbb{R}$$

(a) Show that $\mathrm{f}'(x) = 8(2 + x - x^2)\mathrm{e}^{-2x}$

(3)

a) $f(x) = 4(x^2 - 2)e^{-2x}$

Product Rule

$f(x) = g(x) \cdot h(x)$ then $f'(x) = g'(x)h(x) + g(x)h'(x)$

let $g(x) = 4(x^2 - 2)$ then $g'(x) = 8x$
$h(x) = e^{-2x}$ then $h'(x) = -2e^{-2x}$ 1

$\Rightarrow f'(x) = 8x \cdot e^{-2x} + 4(x^2 - 2) \cdot -2e^{-2x}$ 1
$= 8x \cdot e^{-2x} - 8e^{-2x}(x^2 - 2)$
$f'(x) = 8(x - x^2 + 2)e^{-2x}$

$\Rightarrow f'(x) = 8(2 + x - x^2)e^{-2x}$ as required. 1

(b) Hence find, in simplest form, the exact coordinates of the stationary points of C.

(3)

b) $f'(x) = 8(2+x-x^2)e^{-2x}$

Stationary Points : $f'(x) = 0$

$\Rightarrow 8(2+x-x^2)e^{-2x} = 0$ $\quad \div 8$ and $\div e^{-2x}$ (on both sides)

$\Rightarrow 2+x-x^2 = 0$

$\Rightarrow x^2 - x - 2 = 0$

$\Rightarrow (x-2)(x+1) = 0$

M: -2 A: -1

$-2 + 1 = -1$ ✓

$\Rightarrow x = 2$ and $x = -1$ ①

For $x = 2$, $y = f(2) = 4((2)^2-2)e^{-2(2)} = 4(2)e^{-4} = 8e^{-4} = y$ ①

For $x = -1$, $y = f(-1) = 4((-1)^2-2)e^{-2(-1)} = -4e^2 = y$

$\Rightarrow$ Our coordinates are : $\underline{(2, 8e^{-4})}$ and $\underline{(-1, -4e^2)}$ ①

The function g and the function h are defined by

$$g(x) = 2f(x) \qquad x \in \mathbb{R}$$

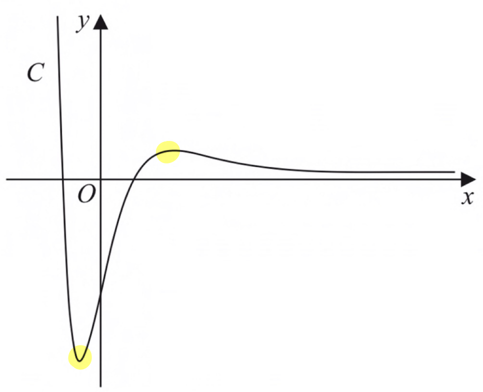
$$h(x) = 2f(x) - 3 \qquad x \geqslant 0$$

(c) Find (i) the range of g

(ii) the range of h

(3)

c) i) $f(x) = 4(x^2-2)e^{-2x} \Rightarrow g(x) = 2f(x) = 8(x^2-2)e^{-2x}$



If coordinates of $f(x) : (a, b)$ then $g(x) : (a, 2b)$

lower limit of range: $2 \times -4e^2 = -8e^2$

upper limit of range: ∞

$\Rightarrow$ Range: $[-8e^2, \infty)$ 1

c) ii) $h(x) = 2f(x) - 3 = 8(x^2-2)e^{-2x} - 3$ for $x \geqslant 0$

The lower limit of the range will be at $x=0 \Rightarrow h(0) = 8(-2)e^{-2\times 0} - 3$

$\Rightarrow h(0) = \underline{-19}$ 1

The upper bound will be our maximum turning point (Since $x \geqslant 0$).
From part b this max turning point had y-value of $8e^{-4}$.
$\Rightarrow$ For the $h(x)$ function this point will be $2 \times 8e^{-4} - 3 = \underline{16e^{-4} - 3}$

Range: $\underline{[-19, 16e^{-4} - 3]}$ 1

10. (a) Use the substitution $x = u^2 + 1$ to show that

$$\int_5^{10} \frac{3\,dx}{(x-1)\left(3+2\sqrt{x-1}\right)} = \int_p^q \frac{6\,du}{u(3+2u)}$$

where p and q are positive constants to be found.

(4)

a) $\displaystyle\int_5^{10} \frac{3}{(\underbrace{x-1}_{u^2+1})(3+2\underbrace{\sqrt{x-1}}_{u})}\,\underbrace{dx}_{2\sqrt{x-1}\,du}$

Integration by Substitution:

$x = u^2 + 1$

$u^2 = x - 1 \Rightarrow u = \sqrt{x-1}$

$du = \frac{1}{2\sqrt{x-1}}\,dx$

$\Rightarrow dx = 2\sqrt{x-1}\,du$ (1)

$= \displaystyle\int_2^3 \frac{3}{(u^2+1-1)(3+2u)} \cdot 2u\,du$ (1)

$= \displaystyle\int_2^3 \frac{6u}{u^2 \cdot (3+2u)}\,du$

New Limits: For $x = 5$, $u = \sqrt{5-1} = \sqrt{4} = 2$
$x = 10$, $u = \sqrt{10-1} = \sqrt{9} = 3$ new limits (1)

$= \displaystyle\int_2^3 \frac{6}{u(3+2u)}\,du$ as required, with $p = 2$ and $q = 3$. (1)

(b) Hence, using algebraic integration, show that

$$\int_5^{10} \frac{3\,dx}{(x-1)\left(3+2\sqrt{x-1}\right)} = \ln a$$

where a is a rational constant to be found.

(6)

b) From part a : $\int_5^{10} \frac{3}{(x-1)(3+2\sqrt{x-1})}\,dx = \int_2^3 \frac{6}{u(3+2u)}\,du$

Partial Fractions : $\frac{6}{u(3+2u)} = \frac{A}{u} + \frac{B}{3+2u}$

$\Rightarrow \quad 6 = A(3+2u) + Bu$

let $u=0 \Rightarrow 6 = 3A \Rightarrow \underline{A = 2}$ and let $u = 1 \Rightarrow 6 = 10 + B$

$\Rightarrow \underline{B = -4}$ 1

$\Rightarrow \int_2^3 \frac{6}{u(3+2u)} = \int_2^3 \frac{2}{u} - \frac{4}{3+2u}$ 1 $du = \left[2\ln(u) - 2\ln(3+2u)\right]_2^3$ 2

$-4\int \frac{1}{3+2u}\,du = \frac{-4\cdot\ln(3+2u)}{2}$

$= -2\ln(3+2u)$

$= \left[2\ln(3) - 2\ln(9)\right] - \left[2\ln(2) - 2\ln(7)\right]$

$\ln(a^b) = b\ln(a)$

$2\ln(9) = 2\ln(3^2) = 4\ln(3)$

$\log(a+b) = \log(ab)$

$\log(a-b) = \log\left(\frac{a}{b}\right)$

$= -2\ln(3) - 2\ln(2) + 2\ln(7)$

$= 2\ln(7/3) - 2\ln(2)$ 1

$= 2\ln\left(\frac{7}{3\times 2}\right) = 2\ln\left(\frac{7}{6}\right) = \ln\left(\frac{7^2}{6^2}\right) = \ln\left(\frac{49}{36}\right) = \ln(a)$ 1

where $a = \underline{\frac{49}{36}}$

11.

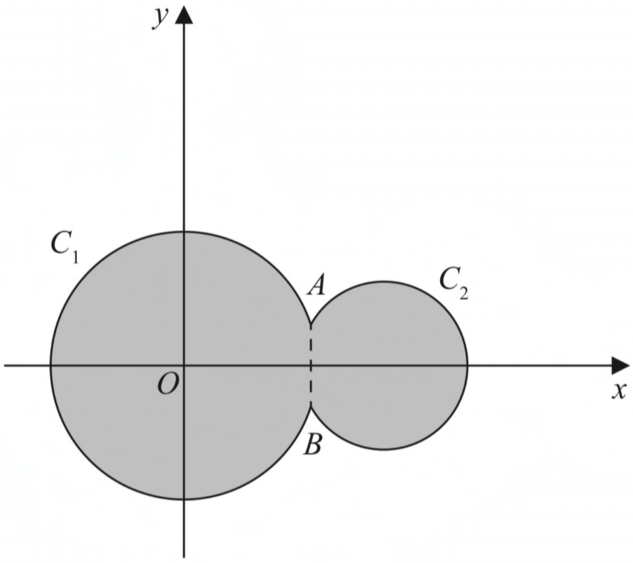


Figure 3

Circle C_1 has equation $x^2 + y^2 = 100$

Circle C_2 has equation $(x - 15)^2 + y^2 = 40$

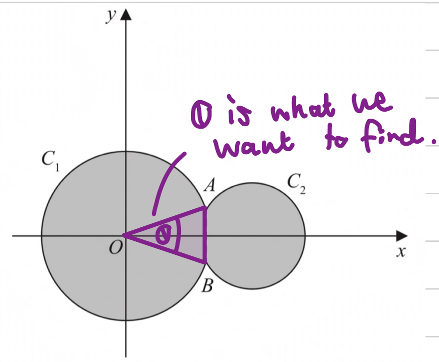
The circles meet at points A and B as shown in Figure 3.

(a) Show that angle $AOB = 0.635$ radians to 3 significant figures, where O is the origin.

(4)

a) $C_1 : x^2 + y^2 = 100$ and $C_2 : (x-15)^2 + y^2 = 40$

$y^2 = 100 - x^2$ (substitute this into C_2)



$\Rightarrow (x-15)^2 + 100 - x^2 = 40$

$x^2 - 30x + 225 + 100 - x^2 = 40$

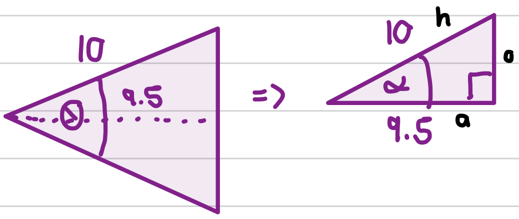
$30x = 285$

$x = \frac{285}{30} = \frac{19}{2}$, or $x = 9.5$ (1). Then $y^2 = 100 - (9.5)^2$

$y^2 = \frac{39}{4} \Rightarrow y = \pm \frac{\sqrt{39}}{2}$

$\Rightarrow y = \pm 3.12$ (1)

$\Rightarrow A : (9.5, 3.12)$ and $B : (9.5, -3.12)$



let $\alpha = \frac{\theta}{2}$ then α : $\cos\alpha = \left(\frac{9.5}{10}\right)$

$\alpha = \cos^{-1}(9.5/10)$

$\alpha = 0.31756$ (1)

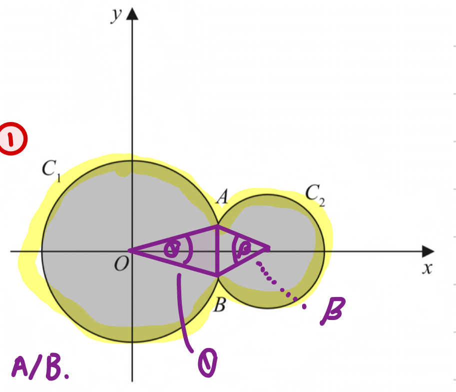
$\Rightarrow \theta = 2\alpha = 2 \times 0.31756 = 0.63512 \Rightarrow$ The angle $AOB = 0.635$ as required. (1)

The region shown shaded in Figure 3 is bounded by C_1 and C_2

(b) Find the perimeter of the shaded region, giving your answer to one decimal place.

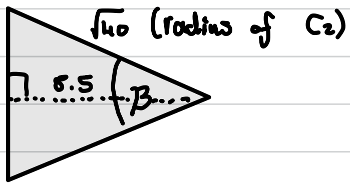
(4)

b) For C_1, we know that $\theta = 0.635$ radians (from part a) and we also know that the radius is 10.



$\Rightarrow$ Perimeter of C_1 (P_1); $P_1 = 10 \times (2\pi - 0.635) = \underline{56.48}$ ①

For C_2:



Centre of C_2

... line: $15 - 9.5 = 5.5$

x coordinate of A/B.

$\beta = 2 \times \cos^{-1}\left(\frac{5.5}{\sqrt{40}}\right) \Rightarrow \beta = 1.03$ radians. ①

$\Rightarrow$ Perimeter of C_2, (P_2) ; P_2 : $\sqrt{40} \times (2\pi - 1.03) = 33.22.$ ①

$\Rightarrow$ Total Perimeter $= P_1 + P_2 = 56.48 + 33.22$

$\Rightarrow$ Total Perimeter $= \underline{89.7}.$ ①

12. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Show that

$$\operatorname{cosec}\theta - \sin\theta \equiv \cos\theta\cot\theta \qquad \theta \neq (180n)^\circ \qquad n \in \mathbb{Z}$$

(3)

a) $\operatorname{Cosec}\theta - \sin\theta \equiv \dfrac{1}{\sin\theta} - \sin\theta$

$\equiv \dfrac{1-\sin^2\theta}{\sin\theta}$ ①

$\equiv \dfrac{\cos^2\theta}{\sin\theta}$

$\equiv \cos\theta \cdot \dfrac{\cos\theta}{\sin\theta}$

$\operatorname{Cosec}\theta - \sin\theta \equiv \cos\theta\cot\theta$ as required. ①

$\operatorname{Cosec}\theta = \dfrac{1}{\sin\theta}$ ①

$\sin^2\theta + \cos^2\theta = 1$

$\Rightarrow 1-\sin^2\theta = \cos^2\theta$

$\dfrac{\cos\theta}{\sin\theta} = \cot\theta$

(b) Hence, or otherwise, solve for $0 < x < 180^\circ$

$$\operatorname{cosec}x - \sin x = \cos x\cot(3x - 50^\circ)$$

(5)

b) Part a : $\operatorname{Cosec}\theta - \sin\theta \equiv \cos\theta\cot\theta$

$\Rightarrow \cos x\cot x = \cos x \cdot \cot(3x-50^\circ)$

$\div\cos x$ $\qquad$ $\div\cos x$

$\Rightarrow \cot x = \cot(3x-50^\circ)$ $\Rightarrow$ $x = 3x - 50$ ①

'equal'

$2x = 50^\circ$

$x = 25^\circ$ ①

Since cot x has a period of 180°, we can find a second solution

$\Rightarrow x + 180 = 3x - 50$ ①

$\Rightarrow 2x = 230^\circ$ $\Rightarrow$ $x = 115^\circ$ ①

There will be a third solution when $\cos x = 0$ $\Rightarrow x = \cos^{-1}(0)$

$\Rightarrow x = 90^\circ$ ①

$\Rightarrow$ $x = 25^\circ$, $x = 90^\circ$ and $x = 115^\circ$

13. A sequence of numbers $a_1, a_2, a_3, \dots$ is defined by

$$a_{n+1} = \frac{k(a_n + 2)}{a_n} \qquad n \in \mathbb{N}$$

where k is a constant.

Given that

- the sequence is a periodic sequence of order 3
- $a_1 = 2$

(a) show that

$$k^2 + k - 2 = 0$$

(3)

a) $a_{n+1} = \dfrac{k(a_n+2)}{a_n}$, What do we know ?

- $a_1 = 2$ — first/initial term
- period of order 3

$\Downarrow$ because of this we know that $\underline{a_4 = a_1}$

Since $a_1 = 2$: $a_2 = \dfrac{k(2+2)}{2} = 2k$ ①

$a_3 = \dfrac{k(2k+2)}{2k} = \dfrac{2k^2+2k}{2k} = k+1$

$a_4 = \dfrac{k(k+1+2)}{k+1} = \dfrac{k(k+3)}{k+1}$

$\Rightarrow a_4 = a_1$ ① $\Rightarrow \dfrac{k(k+3)}{k+1} = 2$

$\Rightarrow k^2+3k = 2k+2 \Rightarrow k^2+k-2 = 0$ as required. ①

(b) For this sequence explain why $k \neq 1$

(1)

b) From part a : $a_1 = 2$, $a_2 = 2k$, $a_3 = k+1$, $a_4 = \frac{k(k+3)}{k+1}$

For $k=1$, we have : $a_1 = 2$, $a_2 = 2$, $a_3 = 2$, $a_4 = 2$

Since all the terms are the same, the sequence no longer has a period of order 3, hence $k \neq 1$ for this sequence. ①

(c) Find the value of

$$\sum_{r=1}^{80} a_r$$

(3)

c) From part a : $k^2 + k - 2 = 0$

$(k-1)(k+2) = 0$

$\Rightarrow k = 1$ and $k = -2$

this is not a valid solution (part b)

$\Rightarrow k = -2.$

$\frac{80}{3} = 26\frac{2}{3}$

$a_1 = 2$

$a_2 = 2k$

$a_3 = k+1$

$a_4 = \frac{k(k+3)}{k+1}$

$\Rightarrow$ $\boxed{\begin{aligned} a_1 &= 2 \\ a_2 &= -4 \\ a_3 &= -1 \end{aligned}}$ ① repeating terms

$a_4 = 2$

$\Rightarrow \sum_{r=1}^{80} a_r = 26 \times (2 - 4 - 1) + 2 - 4$ ①

$= \underline{\underline{-80}}$ ①

14. A large spherical balloon is deflating.

At time t seconds the balloon has radius r cm and volume V cm^3

The volume of the balloon is modelled as decreasing at a constant rate.

(a) Using this model, show that

$$\frac{dr}{dt} = -\frac{k}{r^2}$$

where k is a positive constant.

(3)

a) $\frac{dV}{dt} = -c$ ① (where $c > 0$ is a constant) (we know that the change in volume with respect to time is negative since its decreasing - it's decreasing at a constant rate)

$V = \frac{4}{3}\pi r^3$

$\Rightarrow \underbrace{\frac{dV}{dt}}_{= -c\,*} = \frac{dV}{dr} \times \frac{dr}{dt}$ $\qquad \frac{dV}{dr} = 4\pi r^2\,*$

$\Rightarrow -c = \frac{dr}{dt} \times 4\pi r^2$ ①

$\Rightarrow \frac{dr}{dt} = -\frac{c}{4\pi r^2}$, then let $k = \frac{c}{4\pi}$

$\Rightarrow \frac{dr}{dt} = -\frac{k}{r^2}$ as required. ①

Given that

- the initial radius of the balloon is 40 cm
- after 5 seconds the radius of the balloon is 20 cm
- the volume of the balloon continues to decrease at a constant rate until the balloon is empty

(b) solve the differential equation to find a complete equation linking r and t.

(5)

b) $\frac{dr}{dt} = -\frac{K}{r^2}$ (Solve this using Seperation of Variables)

$\int r^2 \, dr = \int -K \, dt \Rightarrow \boxed{\frac{r^3}{3} = -Kt + \alpha}$ 1 (α is a constant)

Substitute back in !

$t = 0, r = 40 \Rightarrow \frac{40^3}{3} = \boxed{\alpha = \frac{64000}{3}}$ 1

$t = 5, r = 20 \Rightarrow \frac{20^3}{3} = -5K + \frac{64000}{3} \Rightarrow 5K = \frac{56000}{3} \Rightarrow \boxed{K = \frac{11200}{3}}$ 1

$\Rightarrow \frac{r^3}{3} = -\frac{11200}{3} \cdot t + \frac{64000}{3}$

$\Rightarrow r^3 = 64000 - 11200t$ 1

(c) Find the limitation on the values of t for which the equation in part (b) is valid.

(2)

c) $r^3 = 64000 - 11200t$ (equation from part b)

The model will only be valid for non-negative values of r, so we will use this fact to find the limitation on the values of t, where the model is valid.

$64000 - 11200t \geq 0$

$t \leq \frac{64000}{11200} = \frac{40}{7}$ 1

$\Rightarrow$ The model will only be valid for t up to and including $\frac{40}{7}$ seconds. 1

15. The curve C has equation

$$x^2 \tan y = 9 \qquad 0 < y < \frac{\pi}{2}$$

(a) Show that

$$\frac{dy}{dx} = \frac{-18x}{x^4 + 81}$$

(4)

a) We want to use $\boxed{\text{implicit}}$ differentiation to differentiate $x^2 \tan y = 9$

$x^2 \longrightarrow 2x$

$\tan y \longrightarrow \sec^2 y \frac{dy}{dx}$

Product Rule

$h(x) = f(x) \cdot g(x)$ then

$h'(x) = f'(x)\, g(x) + f(x)\, g'(x)$

$\Rightarrow \quad 2x \cdot \tan y + x^2 \sec^2 y \frac{dy}{dx} = 0$ (2)

1 for attempting to differentiate

1 for correct differentiation

We will use the trig identity : $\sec^2 y = 1 + \tan^2 y$ and $\tan y = \frac{9}{x^2}$

$\boxed{\Rightarrow \quad \tan^2 y = \frac{81}{x^4}}$

$\Rightarrow \quad 2x \cdot \frac{9}{x^2} + x^2\left(1 + \frac{81}{x^4}\right)\frac{dy}{dx} = 0$

$\Rightarrow \quad \frac{18}{x} + x^2\left(1 + \frac{81}{x^4}\right)\frac{dy}{dx} = 0$

$\Rightarrow \quad x^2\left(1 + \frac{81}{x^4}\right)\frac{dy}{dx} = -\frac{18}{x} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-\frac{18}{x}}{x^2\left(1 + \frac{81}{x^4}\right)}$ (1) $= \frac{-18}{x^3\left(1 + \frac{81}{x^4}\right)}$ *

* $x^3\left(1 + \frac{81}{x^4}\right) = x^3\left(\frac{x^4 + 81}{x^4}\right) = \frac{x^4 + 81}{x} \longrightarrow \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-18}{\frac{x^4 + 81}{x}}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{-18x}{x^4 + 81}$ as required. (1)

(b) Prove that C has a point of inflection at $x = \sqrt[4]{27}$ $= (27)^{1/4}$

(3)

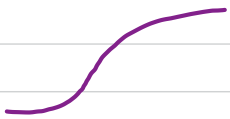
Quotient Rule :

$f(x) = \dfrac{h(x)}{g(x)}$ then

$f'(x) = \dfrac{h'(x) \cdot g(x) - h(x) \cdot g'(x)}{(g(x))^2}$

b) Part a : $\dfrac{dy}{dx} = \dfrac{-18x}{x^4+81}$

Point of inflection :



$-18x \longrightarrow -18$
$x^4+81 \longrightarrow 4x^3$ ①

$\Rightarrow \dfrac{d^2y}{dx^2} = \dfrac{-18(x^4+81) - 4x^3(-18x)}{(x^4+81)^2}$

$= \dfrac{-18x^4 - 1458 + 72x^4}{(x^4+81)^2}$

$= \dfrac{54x^4 - 1458}{(x^4+81)^2} = \dfrac{54(x^4-27)}{(x^4+81)^2} = \dfrac{d^2y}{dx^2}$ ①

$\Rightarrow \dfrac{d^2y}{dx^2} = \dfrac{54(x^4-27)}{(x^4+81)^2}$

- At $x = \sqrt[4]{27} \Rightarrow x^4 = 27 \Rightarrow$ we can substitute this into $\dfrac{d^2y}{dx^2}$

$\Rightarrow$ For $x^4 = 27$, $\dfrac{d^2y}{dx^2} = \dfrac{54(27-27)}{(27+81)^2} = 0$

$\Rightarrow$ For $x^4 > 27$, $\dfrac{d^2y}{dx^2} > 0$

$\Rightarrow$ For $x^4 < 27$, $\dfrac{d^2y}{dx^2} < 0$

$\Rightarrow$ From this we can conclude that there is a point of inflection at $x = \sqrt[4]{27}$. ①

16. Prove by contradiction that there are no positive integers p and q such that

$$4p^2 - q^2 = 25$$

(4)

Proof by Contradiction :
- assume that the first statement is false
- through logical steps, arrive at a conclusion
- deduce that the original statement must be true

$\Rightarrow$ let us assume that there are positive integers p and q such that $4p^2 - q^2 = 25$.

$\Rightarrow$ $4p^2 - q^2 = 25$

$\Rightarrow$ $(2p+q)(2p-q) = 25$ 1

25

$1 \times 25 = 25$

$5 \times 5 = 25$

$\Rightarrow$ Factors are
- 1 and 25
- 5 and 5

$\Rightarrow$ If true then $2p+q = 5$ and $2p - q = 5$ 1

$\Rightarrow$ $q = 5 - 2p$ and $q = 2p - 5$

$\Rightarrow$ $5 - 2p = 2p - 5$

$\Rightarrow$ $4p = 10$ and therefore $p = \underline{2.5}$ not an integer

$\Rightarrow$ $q = 2(2.5) - 5$ $\Rightarrow$ $\underline{q = 0}$ 1

OR If true $2p + q = 25$ and $2p - q = 1$

$\Rightarrow$ $q = 25 - 2p$ and $q = 2p - 1$

$\Rightarrow$ $25 - 2p = 2p - 1$

$\Rightarrow$ $4p = 26$ $\Rightarrow$ $p = \underline{6.5}$ not an integer

$\Rightarrow$ $q = 2(6.5) - 1 = \underline{12}$

OR if true, then $2p + q = 1$ and $2p - q = 25$

$\Rightarrow$ $q = 1 - 2p$ and $q = 2p - 25$

$\Rightarrow$ $1 - 2p = 2p - 25$ $\Rightarrow$ $p = 6.5$ not an integer

$\Rightarrow$ $q = 1 - 2(6.5) = \underline{-12}$ q is not positive.

$\Rightarrow$ This is a contradiction as there are no integer solutions, hence there are no positive integers p and q such that $\underline{4p^2 - q^2 = 25}$. 1