Please check the examination details below before entering your candidate information

Candidate surname MODEL SOLUTIONS	Other names

Pearson Edexcel Level 3 GCE

Centre Number | Candidate Number

Wednesday 12 June 2019

Morning (Time: 2 hours) | Paper Reference **9MA0/02**

Mathematics

Advanced
Paper 2: Pure Mathematics 2

You must have:
Mathematical Formulae and Statistical Tables, calculator

Total Marks

Candidates may use any calculator allowed by Pearson regulations. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need*.
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 14 questions in this question paper. The total mark for this paper is 100.
- The marks for each question are shown in brackets – *use this as a guide as to how much time to spend on each question*.

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over

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Pearson

Answer ALL questions. Write your answers in the spaces provided.

1. Given

$$2^x \times 4^y = \frac{1}{2\sqrt{2}}$$

express y as a function of x.

(3)

$2^x \cdot 4^y = \frac{1}{2\sqrt{2}}$ $\sqrt{2} = 2^{1/2}$ Indice laws:

$2^x \cdot 2^{2y} = \frac{1}{2^1 \cdot 2^{1/2}}$ (1) $a^m \cdot a^n = a^{m+n}$

$1 + \frac{1}{2} = \frac{3}{2}$ $\frac{1}{a^m} = a^{-m}$

$\Rightarrow 2^{x+2y} = \frac{1}{2^{3/2}}$

$a^m = a^n$ then $m = n$

$\Rightarrow 2^{x-2y} = 2^{-3/2}$ (1)

$\Rightarrow x - 2y = -\frac{3}{2}$

$\Rightarrow 2y = -\frac{3}{2} - x$ $\Rightarrow$ $y = -\frac{3}{4} - \frac{x}{2}$ (1)

Question 1 continued

(Total for Question 1 is 3 marks)

2. The speed of a small jet aircraft was measured every 5 seconds, starting from the time it turned onto a runway, until the time when it left the ground.

The results are given in the table below with the time in seconds and the speed in m s^{-1}.

Time (s)	0	5	10	15	20	25
Speed (m s^{-1})	2	5	10	18	28	42

Using all of this information,

(a) estimate the length of runway used by the jet to take off.

(3)

Given that the jet accelerated smoothly in these 25 seconds,

(b) explain whether your answer to part (a) is an underestimate or an overestimate of the length of runway used by the jet to take off.

(1)

a) length of Runway ?

Time (s)	0	5	10	15	20	25
Speed (m s^{-1})	2 y_0	5 y_1	10 y_2	18 y_3	28 y_4	42 y_5

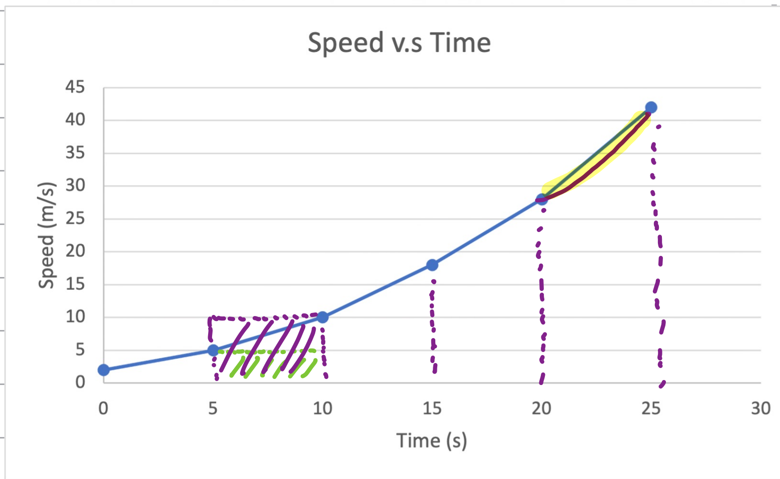
$A = \frac{1}{2} \times h\left[y_0 + y_n + 2(y_1 + \dots + y_{n-1}) \right]$ (1)

$h = 5$ (1)

$A = \frac{1}{2} \times 5\left[2 + 42 + 2(5 + 10 + 18 + 28) \right]$

$\Rightarrow A = 415\text{m}$

$\Rightarrow$ length of runway is 415m. (1)



b) We used the trapezium rule, and we have a concave upwards graph, So we therefore have an overestimate, Since the top of the trapezium is over the curve. (1)

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Question 2 continued

(Total for Question 2 is 4 marks)

3.

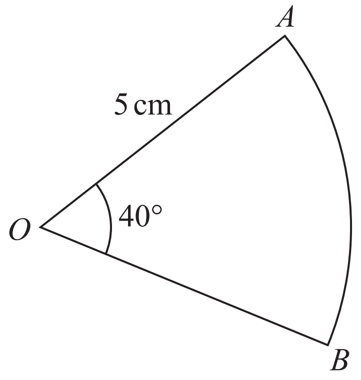


Figure 1

Figure 1 shows a sector AOB of a circle with centre O, radius 5 cm and angle $AOB = 40°$

The attempt of a student to find the area of the sector is shown below.

$$\text{Area of sector} = \frac{1}{2}r^2\theta$$
$$= \frac{1}{2} \times 5^2 \times 40$$
$$= 500\text{ cm}^2$$

(a) Explain the error made by this student.

(1)

(b) Write out a correct solution.

(2)

a) $A = \frac{1}{2}r^2\theta$ is valid for radians, only, so therefore the student uses it incorrectly, since our angle is in degrees (and the student doesn't convert it to radians). (1)

Correct Formula: $A = \frac{\text{Angle}}{360}\pi r^2$

b) Area of the sector $= \frac{\text{angle}}{360}\pi r^2$ angle $= 40°$, $r = 5$cm

$= \frac{40}{360} \times \pi \times 5^2$ (1) $\Rightarrow$ Area of sector $= \frac{25\pi}{9} \approx 8.73\text{ cm}^2$ (1)

Question 3 continued

(Total for Question 3 is 3 marks)

4.

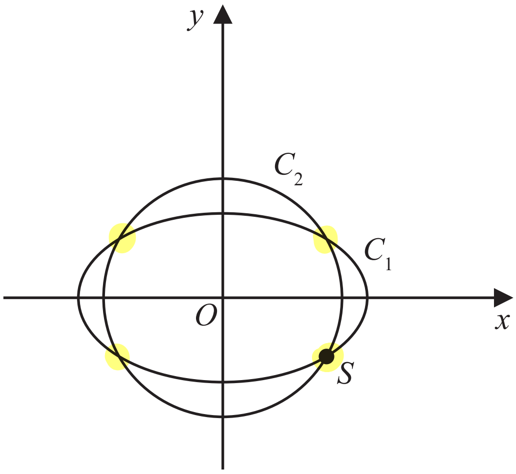


Figure 2

The curve C_1 with parametric equations

$$x = 10\cos t, \quad y = 4\sqrt{2}\sin t, \quad 0 \leqslant t < 2\pi$$

meets the circle C_2 with equation

$$x^2 + y^2 = 66$$

at four distinct points as shown in Figure 2.

Given that one of these points, S, lies in the 4th quadrant, find the Cartesian coordinates of S.

(6)

$C_1 : x = 10\cos t$ and $y = 4\sqrt{2}\sin t$

$C_2 : x^2 + y^2 = 66$

$C_1 \to C_2$

$(4\sqrt{2})^2 = 32$

$\Rightarrow (10\cos t)^2 + (4\sqrt{2}\sin t)^2 = 66$ (1)

$\Rightarrow 100\cos^2 t + 32\sin^2 t = 66$

$\sin^2 t + \cos^2 t = 1 \Rightarrow \cos^2 t = 1 - \sin^2 t$

$\Rightarrow 100(1 - \sin^2 t) + 32\sin^2 t = 66$ (1) (1)

$\Rightarrow 100 - 100\sin^2 t + 32\sin^2 t = 66$

$\Rightarrow 68\sin^2 t = 34$

$\Rightarrow 2\sin^2 t = 1$

$\Rightarrow \sin^2 t = 1/2$

$\Rightarrow \sin t = \frac{1}{\sqrt{2}}$ (1) $\Rightarrow t = \sin^{-1}(1/\sqrt{2}) = \frac{\pi}{4}$

$x = 10\cos t$ $y = 4\sqrt{2}\sin t$

$x = 10\cos(\pi/4)$ $y = 4\sqrt{2}\sin(\pi/4)$

$x = 5\sqrt{2}$ $y = 4 \Rightarrow y = -4$ since S lies below the y-axis. (1)

$\Rightarrow S = (5\sqrt{2}, -4)$ (1)

Question 4 continued

(Total for Question 4 is 6 marks)

5.

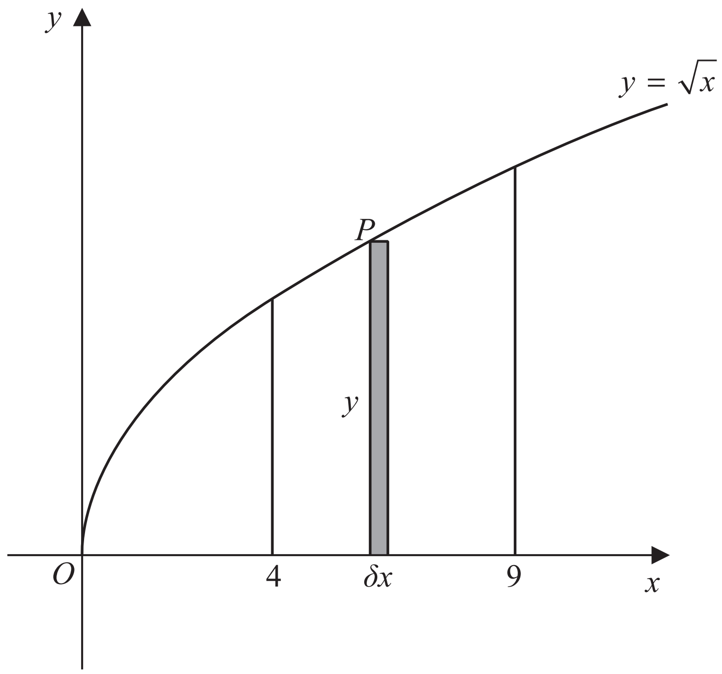


Figure 3

Figure 3 shows a sketch of the curve with equation $y = \sqrt{x}$

The point $P(x, y)$ lies on the curve.

The rectangle, shown shaded on Figure 3, has height y and width δx.

Calculate

$$\lim_{\delta x \to 0} \sum_{x=4}^{9} \sqrt{x}\, \delta x$$

(3)

Since $\delta x \to 0$, $\lim_{\delta x \to 0} \sum_{x=4}^{9} \sqrt{x}\, \delta x = \int_4^9 \sqrt{x}\, dx$ ①

$= \int_4^9 x^{1/2}\, dx = \left[\frac{2x^{3/2}}{3}\right]_4^9$ ①

$\Rightarrow \frac{2(9)^{3/2}}{3} - \frac{2(4)^{3/2}}{3} = \frac{38}{3}$ ①

Question 5 continued

(Total for Question 5 is 3 marks)

6.

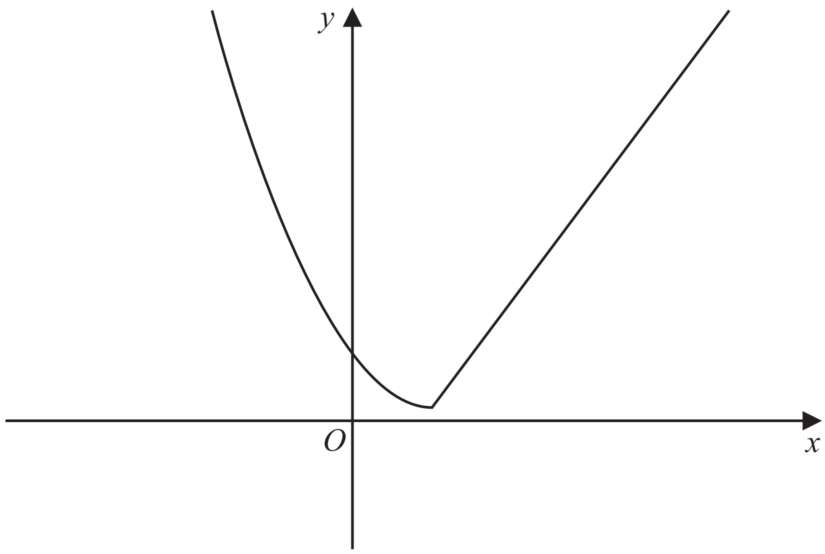


Figure 4

Figure 4 shows a sketch of the graph of $y = \mathrm{g}(x)$, where

$$\mathrm{g}(x) = \begin{cases} (x-2)^2 + 1 & x \leqslant 2 \\ 4x - 7 & x > 2 \end{cases}$$

(a) Find the value of $\mathrm{gg}(0)$.

(2)

(b) Find all values of x for which

$$\mathrm{g}(x) > 28$$

(4)

The function h is defined by

$$\mathrm{h}(x) = (x-2)^2 + 1 \qquad x \leqslant 2$$

(c) Explain why h has an inverse but g does not.

(1)

(d) Solve the equation

$$\mathrm{h}^{-1}(x) = -\frac{1}{2}$$

(3)

Question 6 continued

a) $g(x) = \begin{cases} (x-2)^2 + 1 & x \le 2 \\ 4x-7 & x > 2 \end{cases}$

$g(0) \Rightarrow x = 0 \le 2 \Rightarrow g(x) = (x-2)^2 + 1$

$\Rightarrow g(0) = (0-2)^2 + 1 \Rightarrow g(0) = 5$ (1)

$g(5) \Rightarrow x = 5 > 2 \Rightarrow g(x) = 4x - 7$

$\Rightarrow gg(0) = 4(g(0)) - 7$

$= 4(5) - 7 \Rightarrow g(g(0)) = 13$ (1)

b) $g(x) = \begin{cases} (x-2)^2 + 1 & x \le 2 \\ 4x-7 & x > 2 \end{cases}$

$4x - 7 = 28$ (1) and $(x-2)^2 + 1 = 28$ $\Rightarrow \dfrac{4 \pm \sqrt{(-4)^2 - 4(1)(-23)}}{2}$

$\Rightarrow 4x = 35$ $\quad x^2 - 4x + 5 = 28$

$\Rightarrow x = \dfrac{35}{4}$ (1) $\quad x^2 - 4x - 23 = 0$

↳ quadratic formula $\quad \Rightarrow x = \dfrac{4 \pm \sqrt{108}}{2} = 2 \pm 3\sqrt{3}$ (1) ($\sqrt{108} = 6\sqrt{3}$)

$g\left(\frac{35}{4}\right) = 28$, lets have $g\left(\frac{35}{4} + 0.1\right) = 28.4 > 28 \Rightarrow x > \frac{35}{4}$

$g(2+3\sqrt{3}) = 21.78 \neq 28 \Rightarrow x = 2+3\sqrt{3}$ is not a solution / critical value

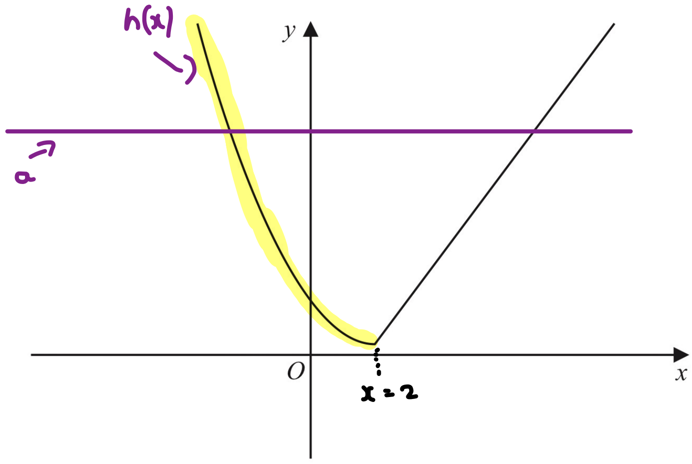
$g(2-3\sqrt{3}) = 28$, then $g(2-0.1-3\sqrt{3}) = 29.05 > 28 \Rightarrow x < 2-3\sqrt{3}$

$\Rightarrow g(x) > 28$ for $x < 2-3\sqrt{3}$ and $x > \frac{35}{4}$ (1)

Question 6 continued

c) h is one-to-one, so it has an inverse by horizontal line test (the line only cuts h(x) once hence one-to-one)

• g is many-to-one, so it does not have an inverse, since the line a intercepts the curve g(x) at more than one point. (1)



d)

$h(x) = (x-2)^2 + 1 \quad x \geq 2$

$f^{-1}(x) = a$ then $x = g(a)$

$h^{-1}(x) = -\frac{1}{2} \Rightarrow x = h(-\frac{1}{2})$ (1)

$\Rightarrow x = (-\frac{1}{2} - 2)^2 + 1 = \frac{29}{4} \Rightarrow x = 7.25$ (1)

(1)

P 5 8 3 5 4 A 0 1 4 4 8

Question 6 continued

(Total for Question 6 is 10 marks)

7. A small factory makes bars of soap.

On any day, the total cost to the factory, £y, of making x bars of soap is modelled to be the sum of two separate elements:

- a fixed cost
- a cost that is proportional to the number of bars of soap that are made that day

(a) Write down a general equation linking y with x, for this model. **(1)**

The bars of soap are sold for £2 each.

On a day when 800 bars of soap are made and sold, the factory makes a profit of £500

On a day when 300 bars of soap are made and sold, the factory makes a loss of £80

Using the above information,

(b) show that $y = 0.84x + 428$ **(3)**

(c) With reference to the model, interpret the significance of the value 0.84 in the equation. **(1)**

Assuming that each bar of soap is sold on the day it is made,

(d) find the least number of bars of soap that must be made on any given day for the factory to make a profit that day. **(2)**

a)

$y = kx + c$ } fixed ①

k: proportional constant.

Fixed Cost => Constant

proportional Cost => related to number of bars made

b) i) $y = 2(800) - 500 = 1100 \Rightarrow (x, y) = (800, 1100)$ * $y = kx + c$ *

ii) $y = 2(300) + 80 = 680 \Rightarrow (x, y) = (300, 680)$ ①

$\Rightarrow$ i) $1100 = 800k + c$ and for ii) $680 = 300k + c$

$c = 1100 - 800k$ and $c = 680 - 300k$

$\Rightarrow 1100 - 800k = 680 - 300k \Rightarrow 500k = 420 \Rightarrow k = 0.84$

$\Rightarrow c = 1100 - 800(0.84) = 428 = c$ ②

$\Rightarrow y = 0.84x + 428$ as required. ③

P 5 8 3 5 4 A 0 1 6 4 8

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Question 7 continued

c) $y = 0.84x + 428$

$\Rightarrow$ 0.84 is the cost of making each additional/extra bar of soap. (1)

d) letting n be the least number of bars required to make a profit.

Then $2n = 0.84n + 428$ (1)

$\Rightarrow n = 0.42n + 214 \Rightarrow 0.58n = 214$

$\Rightarrow n = 368.965...$

$\Rightarrow n = \underline{369}$ bars (1)

Question 7 continued

Question 7 continued

(Total for Question 7 is 7 marks)

8. (i) Find the value of

$$\sum_{r=4}^{\infty} 20 \times \left(\frac{1}{2}\right)^r$$

(3)

(ii) Show that

$$\sum_{n=1}^{48} \log_5\left(\frac{n+2}{n+1}\right) = 2$$

(3)

i) $\sum_{r=4}^{\infty} 20 \times \left(\frac{1}{2}\right)^r$ $S_\infty = \sum^{\infty} = \frac{a}{1-r}$

$= 20\left(\frac{1}{2}\right)^4 + 20\left(\frac{1}{2}\right)^5 + 20\left(\frac{1}{2}\right)^6 + \ldots$

a : first term

r : common ratio

$a = 20\left(\frac{1}{2}\right)^4$, $r = \frac{1}{2}$ ①

$\Rightarrow \sum_{r=4}^{\infty} 20 \cdot \left(\frac{1}{2}\right)^r = \frac{20\left(\frac{1}{2}\right)^4}{1-\frac{1}{2}}$ ② $= 5/2 = 2.5$ ③

ii) $\sum_{n=1}^{48} \log_5\left(\frac{n+2}{n+1}\right)$ $\log\left(\frac{a}{b}\right) = \log(a) - \log(b)$

$\Rightarrow \sum_{n=1}^{48} \log_5(n+2) - \log_5(n+1)$ ①

$= \left[\log_5(3) + \log_5(4) + \ldots + \log_5(50)\right] - \left[\log_5(2) + \log_5(3) + \ldots + \log_5(49)\right]$ ②

$= \log_5(50) - \log_5(2) = \log_5\left(\frac{50}{2}\right) = \log_5(25) = 2$

$\Rightarrow \sum_{n=1}^{48} \log_5\left(\frac{n+2}{n+1}\right) = 2$ as required. ③

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Question 8 continued

Question 8 continued

Question 8 continued

(Total for Question 8 is 6 marks)

A research engineer is testing the effectiveness of the braking system of a car when it is driven in wet conditions.

The engineer measures and records the braking distance, d metres, when the brakes are applied from a speed of $V \text{ km h}^{-1}$.

Graphs of d against V and $\log_{10} d$ against $\log_{10} V$ were plotted.

The results are shown below together with a data point from each graph.

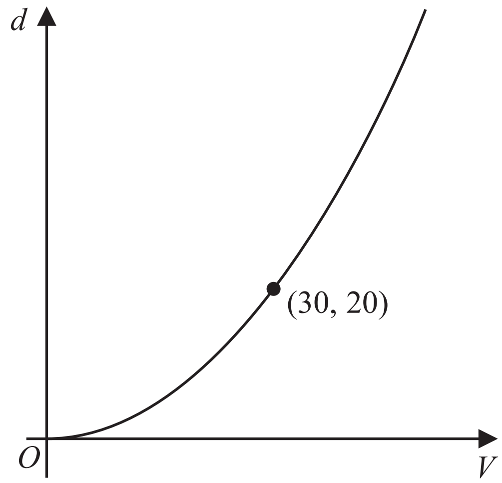

Figure 5

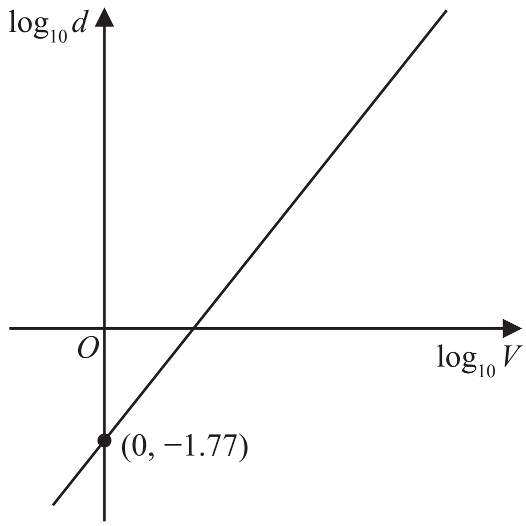

Figure 6

(a) Explain how Figure 6 would lead the engineer to believe that the braking distance should be modelled by the formula

$$d = kV^n \qquad \text{where } k \text{ and } n \text{ are constants}$$

with $k \approx 0.017$

(3)

Using the information given in Figure 5, with $k = 0.017$

(b) find a complete equation for the model giving the value of n to 3 significant figures.

(3)

Sean is driving this car at 60 km h^{-1} in wet conditions when he notices a large puddle in the road 100 m ahead. It takes him 0.8 seconds to react before applying the brakes.

(c) Use your formula to find out if Sean will be able to stop before reaching the puddle.

(3)

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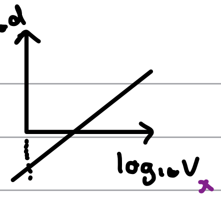
Question 9 continued

a) $d = kV^n$, $k \approx 0.017$



Log laws :
$\log(ab) = \log(a) + \log(b)$
$\log(a^m) = m\log(a)$

$\log_{10}(d) = \log_{10}(kV^n)$
$= \log_{10}(k) + \log_{10}(V^n)$
$\log_{10}(d) = \log_{10}(k) + n\log_{10} V$

$k \approx 0.017$, $\log_{10}(0.017) = -1.7695...$ (1)

$\Rightarrow \log_{10}(d) = \log_{10}(0.017) + n\log_{10} V$
$\Rightarrow \log_{10}(d) = n\log_{10} V - 1.77$ (2)
$y = mx + c$

$\Rightarrow$ The linear nature and y-intercept in the equation matching that of figure 6 tells us that braking distance can be modelled by $d = kV^n$. (3)

b)
$d = 0.017 \cdot V^n$
For $(30, 20) \Rightarrow 20 = 0.017 \cdot 30^n$ (1)
$\Rightarrow 30^n = \dfrac{20}{0.017}$

$\log(a^m) = m\log(a)$

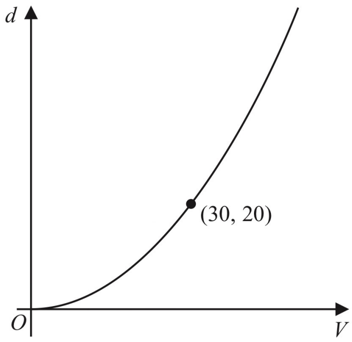


Figure 5

$\Rightarrow \log_{10}(30^n) = \log_{10}\left(\dfrac{20}{0.017}\right)$

$\Rightarrow n \cdot \log_{10}(30) = \log_{10}(20/0.017)$

$\Rightarrow n = \dfrac{\log_{10}\left(\frac{20}{0.017}\right)}{\log_{10}(30)} = 2.07876...$

$\Rightarrow n = 2.08$ (1) $\Rightarrow d = 0.017 \cdot V^{2.08}$ (1)

c) $d = 0.017 \cdot V^{2.08}$

Distance : before + after breaks

$0.8s = \dfrac{0.8}{3600}$ hrs

Before brakes: d_1 = Speed $\times$ time $= 60 \times \dfrac{0.8}{3600} = 0.0133...$ km
$\Rightarrow d_1 = 13.3$m (1)

$d_2 = 0.017 \cdot 60^{2.08}$ (1)
$d_2 = 84.9$m

$\Rightarrow$ Total distance $= 84.9 + 13.3 = 98.2$m $\Rightarrow$ Sean stops in time. (1)

Question 9 continued

Question 9 continued

(Total for Question 9 is 9 marks)

10.

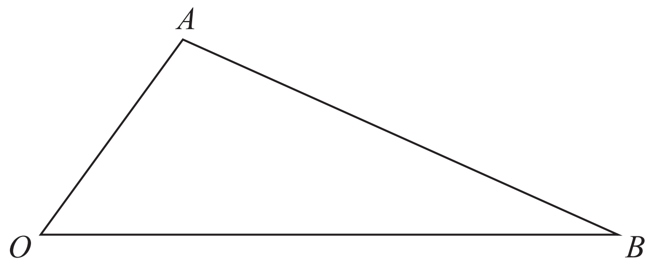


Figure 7

Figure 7 shows a sketch of triangle OAB.

The point C is such that $\overrightarrow{OC} = 2\overrightarrow{OA}$.

The point M is the midpoint of AB.

The straight line through C and M cuts OB at the point N.

Given $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$

(a) Find $\overrightarrow{CM}$ in terms of $\mathbf{a}$ and $\mathbf{b}$ **(2)**

(b) Show that $\overrightarrow{ON} = \left(2 - \frac{3}{2}\lambda\right)\mathbf{a} + \frac{1}{2}\lambda\mathbf{b}$, where λ is a scalar constant. **(2)**

(c) Hence prove that $ON : NB = 2 : 1$ **(2)**

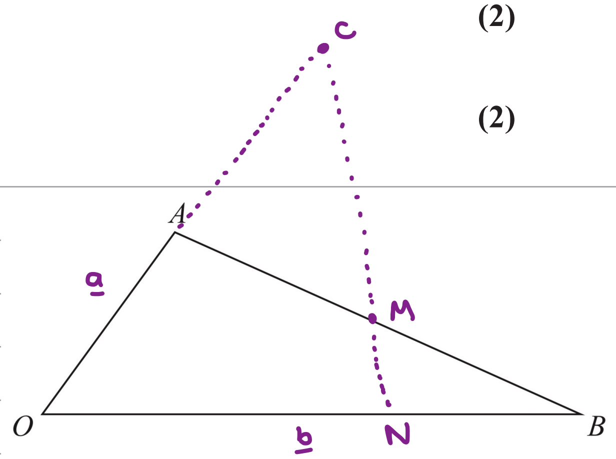
a) $\overrightarrow{CM} \leq \frac{a}{b}$?

$\Rightarrow \overrightarrow{CM} = \overrightarrow{CA} + \overrightarrow{AM}$

$\Rightarrow \overrightarrow{CM} = -\underline{a} + \frac{1}{2}(-\underline{a} + \underline{b})$

$\overrightarrow{CA} = -\overrightarrow{OA} = -\underline{a}$

$\overrightarrow{AM} = \frac{1}{2}(-a + b)$



$\Rightarrow \overrightarrow{CM} = -\underline{a} - \frac{1}{2}\underline{a} + \frac{1}{2}\underline{b}$ (1)

$\Rightarrow \overrightarrow{CM} = -\frac{3}{2}\underline{a} + \frac{1}{2}\underline{b}$ (1)

Question 10 continued

$\vec{ON} = \vec{OC} + \vec{CN}$

$\vec{OC} = 2\underline{a}$

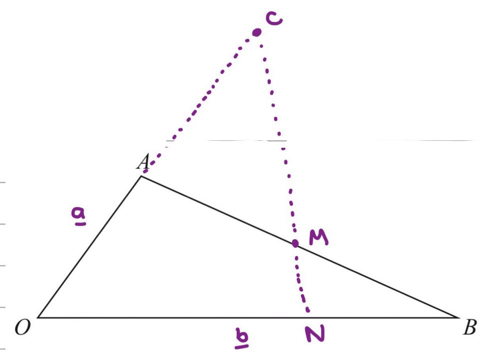
$\vec{CN} = \lambda\vec{CM}$ ①

$= \lambda\left(-\frac{3}{2}\underline{a} + \frac{1}{2}\underline{b}\right)$

$\vec{ON} = 2\underline{a} + \lambda\left(-\frac{3}{2}\underline{a} + \frac{1}{2}\underline{b}\right)$

$\vec{ON} = 2\underline{a} - \frac{3}{2}\lambda\underline{a} + \frac{1}{2}\lambda\underline{b}$

$\vec{ON} = \left(2 - \frac{3}{2}\lambda\right)\underline{a} + \frac{1}{2}\lambda\underline{b}$ as required. ①



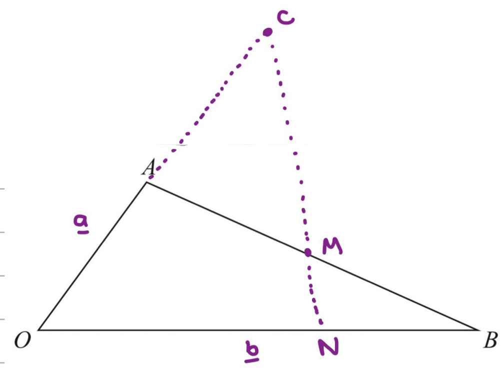
c)

$\vec{NB} = 2\vec{ON}$

$\left(2 - \frac{3}{2}\lambda\right) = 0 \Rightarrow 4 = 3\lambda \Rightarrow \lambda = \frac{4}{3}$ ①

$\Rightarrow \vec{ON} = \frac{2}{3}\underline{b} \Rightarrow \vec{NB} = \frac{1}{3}\underline{b}$

$\Rightarrow \vec{ON} : \vec{NB} = 2:1$ as required. ①



Question 10 continued

Question 10 continued

(Total for Question 10 is 6 marks)

11.

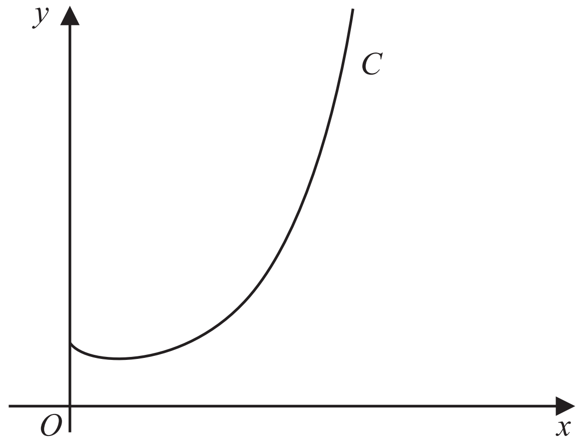


Figure 8

Figure 8 shows a sketch of the curve C with equation $y = x^x$, $x > 0$

(a) Find, by firstly taking logarithms, the x coordinate of the turning point of C.

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(5)

The point $P(\alpha, 2)$ lies on C.

(b) Show that $1.5 < \alpha < 1.6$

(2)

A possible iteration formula that could be used in an attempt to find α is

$$x_{n+1} = 2x_n^{\,1-x_n}$$

Using this formula with $x_1 = 1.5$

(c) find x_4 to 3 decimal places,

(2)

(d) describe the long-term behaviour of x_n

(2)

a) $y = x^x$

$\ln y = \ln(x^x)$

$\Rightarrow \ln y = x \cdot \ln(x)$ ①

$\frac{1}{y}\frac{dy}{dx} = x \cdot \frac{1}{x} + \ln(x) \cdot 1$

$\frac{1}{y}\frac{dy}{dx} = 1 + \ln(x)$ ②

$\frac{dy}{dx} = 0 \Rightarrow \frac{1}{y} \cdot 0 = 1 + \ln x$

$\Rightarrow 0 = 1 + \ln x$

$\Rightarrow \ln(x) = -1$

$\Rightarrow e^{\ln x} = e^{-1} \Rightarrow x = \frac{1}{e} = 0.368$ ①

①

Turning Point ?

↳ $\frac{dy}{dx} = 0$

log laws : $\ln(a^m) = m \cdot \ln(a)$

Product Rule : $h(x) = f(x) \cdot g(x)$

$h'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

$x \rightarrow 1$

$\ln x \rightarrow \frac{1}{x}$

DO NOT WRITE IN THIS AREA

Question 11 continued

b) $y = x^x$

$x = 1.5 \Rightarrow y = 1.5^{1.5} = 1.84$

$x = 1.6 \Rightarrow y = 1.6^{1.6} = 2.12$ (1)

$P(\alpha, 2) \Rightarrow 1.84 < 2 < 2.12$, we also know that C is a continuous curve, hence $1.5 < \alpha < 1.6$ (1)

c) $x_{n+1} = 2x_n^{1-x_n}$, $x_1 = 1.5$

$x_2 = 2 \cdot x_1^{1-x_1} = 2 \cdot (1.5)^{1-1.5} = 1.63299..$ (1)

$x_3 = 2 \cdot x_2^{1-x_2} = 1.46626...$

$x_4 = 2 \cdot x_3^{1-x_3} = 1.6731... \Rightarrow x_4 = 1.673$ (1)

d) $n \to \infty$, what happens to x_n?

- x_n fluctuates between 1 and 2 (1) 1, 2, 1, 2, ...
- x_n will be periodic with period 2 (1)

Question 11 continued

Question 11 continued

(Total for Question 11 is 11 marks)

Turn over

12. (a) Prove

$$\frac{\cos 3\theta}{\sin\theta} + \frac{\sin 3\theta}{\cos\theta} \equiv 2\cot 2\theta \qquad \theta \neq (90n)°,\ n \in \mathbb{Z}$$

(4)

(b) Hence solve, for $90° < \theta < 180°$, the equation

$$\frac{\cos 3\theta}{\sin\theta} + \frac{\sin 3\theta}{\cos\theta} = 4$$

giving any solutions to one decimal place.

(3)

a) $\frac{\cos 3\theta}{\sin\theta} + \frac{\sin 3\theta}{\cos\theta} \equiv 2\cot 2\theta$

Trig Identities:

- $\cos(a-b) = \cos a\cos b + \sin a \sin b$
  ↳ $a = 3\theta$, $b = \theta$

$\Rightarrow \frac{\cos 3\theta\cos\theta + \sin 3\theta\sin\theta}{\sin\theta\cos\theta}$ (1)

$\Rightarrow \frac{\cos(3\theta-\theta)}{\sin\theta\cos\theta} = \frac{\cos(2\theta)}{\sin\theta\cos\theta}$ (1)

- $\sin 2\theta = 2\sin\theta\cos\theta$
  $\Rightarrow \frac{1}{2}\sin 2\theta = \sin\theta\cos\theta$
- $\frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta} = \cot\theta$

$= \frac{\cos(2\theta)}{\frac{1}{2}\sin(2\theta)} = \frac{2}{\tan 2\theta}$ (1)

$= 2\cot 2\theta$

$\Rightarrow \frac{\cos 3\theta}{\sin\theta} + \frac{\sin 3\theta}{\cos\theta} \equiv 2\cot 2\theta$ as required. (1)

b) $\frac{\cos 3\theta}{\sin\theta} + \frac{\sin 3\theta}{\cos\theta} = 4$

Part a: $\frac{\cos 3\theta}{\sin\theta} + \frac{\sin 3\theta}{\cos\theta} = 2\cot 2\theta = \frac{2}{\tan 2\theta}$

S | A
T | C

$\Rightarrow 2\cot 2\theta = 4 \Rightarrow \frac{2}{\tan 2\theta} = 4 \Rightarrow \tan 2\theta = \frac{1}{2}$ (1)

$\Rightarrow 2\theta = \tan^{-1}\left(\frac{1}{2}\right)$ (1) ↙ $180 + 26.6$

$\Rightarrow 2\theta = 26.6°$ and $2\theta = 206.6°$

$\Rightarrow \theta = 13.3°$ and $\theta = 103.3°$

$\Rightarrow 13.3°$ is not a valid solution since out of range $\Rightarrow \theta = 103.3°$ (1)

P 5 8 3 5 4 A 0 3 6 4 8

Question 12 continued

Question 12 continued

Question 12 continued

(Total for Question 12 is 7 marks)

13.

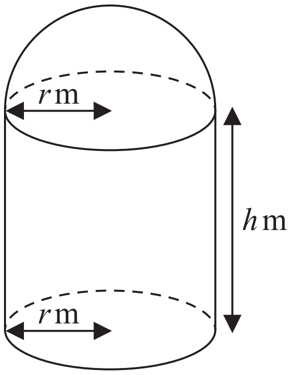


Figure 9

[A sphere of radius r has volume $\frac{4}{3}\pi r^3$ and surface area $4\pi r^2$]

A manufacturer produces a storage tank.

The tank is modelled in the shape of a hollow circular cylinder closed at one end with a hemispherical shell at the other end as shown in Figure 9.

The walls of the tank are assumed to have negligible thickness.

The cylinder has radius r metres and height h metres and the hemisphere has radius r metres.

The volume of the tank is $6\,\text{m}^3$.

(a) Show that, according to the model, the surface area of the tank, in m^2, is given by

$$\frac{12}{r} + \frac{5}{3}\pi r^2$$

(4)

The manufacturer needs to minimise the surface area of the tank.

(b) Use calculus to find the radius of the tank for which the surface area is a minimum.

(4)

(c) Calculate the minimum surface area of the tank, giving your answer to the nearest integer.

(2)

a) $A = A_1 + A_2 + A_3$

Area Cylinder : $A_1 = 2\pi rh$

Area base : $A_2 = \pi r^2$

Area hemisphere : $A_3 = 2\pi r^2$

$V = 6m^3 = V_{cylinder} + V_{hsphere}$

$6 = \pi r^2 h + \frac{2\pi r^3}{3}$ (1) $\Rightarrow \pi r^2\left(h + \frac{2}{3}r\right) = 6$

$\Rightarrow h = \frac{6}{\pi r^2} - \frac{2}{3}r$ (1)

$\Rightarrow A = 2\pi rh + \pi r^2 + 2\pi r^2$ (1)

$\Rightarrow A = 2\pi rh + 3\pi r^2$

$\Rightarrow A = \left(\frac{6}{\pi r^2} - \frac{2}{3}r\right)2\pi r + 3\pi r^2$

$\Rightarrow A = \frac{12\pi r}{\pi r^2} - \frac{4}{3}\pi r^2 + 3\pi r^2$

$\Rightarrow A = \frac{12}{r} + \frac{5}{3}\pi r^2$ (1) as required.

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

Question 13 continued

b) $A = \frac{12}{r} + \frac{5}{3}\pi r^2$

① Differentiate
② Set equal 0
③ Solve for r

$\frac{dA}{dr} = -\frac{12}{r^2} + \frac{10\pi r}{3} = 0$ ②

$\Rightarrow -\frac{36}{r^2} + 10\pi r = 0$

$\Rightarrow -36 + 10\pi r^3 = 0 \Rightarrow 10\pi r^3 = 36$

$\Rightarrow r^3 = \frac{36}{10\pi}$ ① $\Rightarrow r = \sqrt[3]{\frac{36}{10\pi}} = 1.05\text{m}$

$\Rightarrow$ Surface area will be a minimum when $r = 1.05\text{m}$ ①

c) $A = \frac{12}{r} + \frac{5}{3}\pi r^2$, $r = 1.05\text{m}$

$A = \frac{12}{1.05} + \frac{5}{3}\pi(1.05)^2 = 17.201\ldots$ ①

$\Rightarrow$ Minimum Surface is 17m^2 ①

Question 13 continued

Question 13 continued

(Total for Question 13 is 10 marks)

14. (a) Use the substitution $u = 4 - \sqrt{h}$ to show that

$$\int \frac{\mathrm{d}h}{4-\sqrt{h}} = -8\ln|4-\sqrt{h}| - 2\sqrt{h} + k$$

where k is a constant

(6)

A team of scientists is studying a species of slow growing tree.

The rate of change in height of a tree in this species is modelled by the differential equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{t^{0.25}\left(4-\sqrt{h}\right)}{20}$$

where h is the height in metres and t is the time, measured in years, after the tree is planted.

(b) Find, according to the model, the range in heights of trees in this species.

(2)

One of these trees is one metre high when it is first planted.

According to the model,

(c) calculate the time this tree would take to reach a height of 12 metres, giving your answer to 3 significant figures.

(7)

a) $\int \frac{1}{4-\sqrt{h}}\,dh$

$u = 4-\sqrt{h} \Rightarrow \sqrt{h} = 4-u$

$du = -\frac{1}{2\sqrt{h}}\,dh \Rightarrow dh = -2\sqrt{h}\,du \Rightarrow dh = -2(4-u)\,du \Rightarrow dh = -8+2u\,du$ ①

$\Rightarrow \int \frac{1}{4-\sqrt{h}}\,dh = \int \frac{1}{4-(4-u)}(-8+2u)\,du = \int \frac{2u-8}{u}\,du = \int \frac{2u}{u} - \frac{8}{u}\,du$ ①

$= \int 2 - \frac{8}{u}\,du = 2u - 8\ln u + c$ ① ②

$= 2(4-\sqrt{h}) - 8\ln|4-\sqrt{h}| + c$

$\Rightarrow 8 - 2\sqrt{h} - 8\ln|4-\sqrt{h}| + c$ $\quad c+8 = k$

$\Rightarrow \int \frac{1}{4-\sqrt{h}}\,dh = -8\ln|4-\sqrt{h}| - 2\sqrt{h} + k$ as required ①

DO NOT WRITE IN THIS AREA

Question 14 continued

b) $\frac{dh}{dt} = \frac{t^{0.25}(4-\sqrt{h})}{20}$

• When the trees stop growing, $\frac{dh}{dt}$ (rate of change of height) is going to be equal to zero.

$\frac{dh}{dt} = \frac{t^{0.25}(4-\sqrt{h})}{20} = 0 \Rightarrow t^{0.25}(4-\sqrt{h}) = 0$

$\Rightarrow 4-\sqrt{h} = 0$ (1)

$\Rightarrow \sqrt{h} = 4$

$\Rightarrow h = 16\text{m}$

$\Rightarrow$ Range of heights of trees will be $0 \le h \le 16\text{m}$ (1)

c) $\frac{dh}{dt} = \frac{t^{0.25}(4-\sqrt{h})}{20}$ and $t=0, h=1\text{m} \Rightarrow t = ?$ when $h = 12\text{m}$

$\int \frac{1}{4-\sqrt{h}}\,dh = \int \frac{t^{0.25}}{20}\,dt$ (1)

Part a

$\int \frac{t^{0.25}}{20}\,dt = \frac{t^{1.25}}{1.25\times 20} = \frac{t^{1.25}}{25} + C$

$\Rightarrow -8\ln|4-\sqrt{h}| - 2\sqrt{h} = \frac{t^{1.25}}{25}\ (+c)$ (2)

Substitute back in!

$\Rightarrow t = 0$ and $h = 1 \Rightarrow -8\ln|4-\sqrt{1}| - 2\sqrt{1} = C$ (1)

$\Rightarrow -8\ln(3) - 2 = c$

$\Rightarrow -8\ln|4-\sqrt{h}| - 2\sqrt{h} = \frac{t^{1.25}}{25} - 8\ln(3) - 2$ (1)

$\Rightarrow h = 12\text{m}, t = ? \Rightarrow -8\ln|4-\sqrt{12}| - 2\sqrt{12} = \frac{t^{1.25}}{25} - 8\ln(3) - 2$

$\Rightarrow \sqrt[1.25]{(-8\ln|4-\sqrt{12}| - 2\sqrt{12} + 8\ln(3) + 2)\,25} = t^{1.25}$ (1)

$\Rightarrow t = 75.2$ years when $h = 12\text{m}$ (1)

Question 14 continued

Question 14 continued

Question 14 continued

(Total for Question 14 is 15 marks)

TOTAL FOR PAPER IS 100 MARKS